DIFFERENTIAL GEOMETRY AND ITS APPLICATIONS
Proc. Conf. Prague, August 30 – September 3, 2004
Charles University, Prague (Czech Republic), 2005, 45 – 58

Stability, integral invariants and canonical Kähler metrics

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Abstract. In this expository article we will summarize recent results on stability and Kähler manifolds of constant scalar curvature. We will see among other things that there is a family of integral invariants which includes obstructions to the existence of Kähler metrics of harmonic Chern forms and obstructions to asymptotic Chow semistability.

1. Introduction

In this expository article we discuss on the relationship between the existence of canonical Kähler metrics on compact Kähler manifolds and stability in the sense of geometric invariant theory. Typical canonical Kähler metrics are Kähler-Einstein metrics. As is well-known the existence of Kähler-Einstein metrics was proved by Aubin [**1**] and Yau [**31**] in the negative first Chern class case and by Yau [**31**] in the zero first Chern class case. On the other hand there are obstructions to the existence of Kähler-Einstein metrics in the positive case which are described in terms of the complex Lie algebra of all holomorphic vector fields ([**23**], [**13**]). These results are extended to obstructions to the existence of Kähler metrics of constant scalar curvature. More recently new types of obstructions, which are unrelated to the Lie algebra of holomorphic vector fields, are proved to exist. Tian [**30**] first introduced a notion called K-stability considering the degeneration of Fano manifolds to normal varieties and using an integral invariant by Ding and Tian [**6**] which generalizes the author's invariant for non-singular Fano varieties to the normal varieties. Then Donaldson proved in [**8**] that the existence of constant scalar curvature metrics on a polarized manifold implies the asymptotic Chow stability of the manifold when there are no nontrivial holomorphic vector fields.

In view of these two works of Tian and Donaldson it is reasonable to expect that the stability would be the right answer to the problem of finding an intrinsic condition for the existence of canonical Kähler metrics. In [**9**] Donaldson re-defines the notion of K-stability by considering degenerations to non-normal varieties, and conjectures that the existence of a constant scalar curvature Kähler metric on a polarized manifold would be equivalent to the K-stability of the polarized manifold. In this article we will see the relationship between the existence of constant scalar curvature metrics and various notions of stability when the Lie algebra of holomorphic vector fields is nontrivial. In fact in such a case there are aspects where we

2000 *Mathematics Subject Classification.* Primary 53C55, Secondary 53C21, 55N91 .
Key words and phrases. stability, constant scalar curvature, Kähler manifold.
This paper is in final form and no version of it will be published elswhere.

can see clearly the relationship between the existence of constant scalar curvature metrics and stability. This article consists of three parts. The first part (section 2) is written based on the author's paper [**15**] in which we see that there is a family of integral invariants which includes obstructions for a Kähler class to contain a Kähler metric of harmonic Chern form and also obstructions for a polarized manifold to be asymptotically Chow semistable. The second part (section 3) is based on papers by Fujiki [**12**], Donaldson [**7**] and Wang [**32**] in which they set up a moment map in an infinite dimensional setting and the zero set corresponds to the set of Kähler metrics of constant scalar curvature. Recall that for a Hamiltonian action of a compact group K on a compact Kähler manifold, having a zero of the moment map along the orbit of the complexified group K^c-action is equivalent to the stability of the orbit of the reductive group K^c (c.f. [**11**], section 6.5). We see in this setting that the obstructions related to holomorphic vector fields can be obtained by formally applying some simple results on finite dimensional moment maps. The third part (section 4) is based on the papers of Ross and Thomas ([**27**], [**28**]) in which they compare the K-stability, Chow stability and Hilbert stability, and introduce a notion called slope stability.

2. Integral invariants and asymptotic Chow semistability

A Kähler metric $g = (g_{i\bar{j}})$ is said to be an extremal Kähler metric if the (1,0)-part of the gradient vector field

$$grad^{1,0}\sigma = \sum_{i,j=1}^{m} g^{i\bar{j}} \frac{\partial \sigma}{\partial z^{\bar{j}}} \frac{\partial}{\partial z^i}$$

of the scalar curvature σ is a holomorphic vector field. Such a metric is a critical point of the functional

$$g \mapsto \int_M |\sigma|^2 dV_g$$

on the space of metrics in a fixed Kähler class. If the scalar curvature σ is constant then its gradient vector field is 0. Therefore a Kähler metric of constant scalar curvature is an extremal Kähler metric. A Kähler-Einstein metric is a Kähler metric whose Ricci curvature

$$R_{i\bar{j}} = -\frac{\partial^2}{\partial z^i \partial \overline{z}^j} \log \det g,$$

is proportional to g. Thus for some constant k

$$R_{i\bar{j}} = k g_{i\bar{j}}. \tag{1}$$

Such a metric has constant scalar curvature, and of course is a special case of extremal Kähler metric. On the other hand, since the Ricci form

$$\rho_g = \frac{i}{2\pi} \sum_{i=1}^{m} R_{i\bar{j}} dz^i \wedge d\bar{z}^j$$

represents the first Chern class $c_1(M)$ as a de Rham class, according as the sign of k in (1) is positive, zero or negative, $c_1(M)$ is represented by a positive definite, zero or negative definite (1.1)-form. Such situations are expressed by saying that $c_1(M) > 0$, $c_1(M) = 0$ or $c_1(M) < 0$. Namely in order for M to admit a Kähler-Einstein metric one of these three conditions has to hold. As is mentioned in the introduction the remaining case to be solved is the positive case.

As is indicated in the variational formulation of extremal Kähler metrics we fix a de Rham class $[\omega_0]$ of a fixed Kähler form ω_0. Choose any Kähler form $\omega \in [\omega_0]$. Denote by $\mathfrak{h}(M)$ the complex Lie algebra of all holomorphic vector fields, and put

$$\mathfrak{h}_0(M) = \{X \in \mathfrak{h}(M) \,|\, X \text{ has a zero}\}.$$

Then it is well-known that for $X \in \mathfrak{h}_0(M)$ there is a smooth complex valued function u_X uniquely up to a constant such that

$$i(X)\omega = -\overline{\partial} u_X.$$

In this sense $\mathfrak{h}_0(M)$ coincides with the set of all holomorphic Hamiltonian vector fields. We may normalize u_X so that

$$\int_M u_X\, \omega^m = 0. \tag{2}$$

Let $P_G \to M$ be a holomorphic principal bundle where a complex Lie G acts from the right as the structure group and a complex Lie group H acts from the left as bundle maps commuting with the action of G. Therefore H also acts on M as biholomorphic automorphisms.

Let θ be a type $(1,0)$-connection of the principal bundle P_G. This means that the connection form in P_G is type $(1,0)$. Typical examples of such connections are those of the frame bundles of Hermitian holomorphic vector bundles, say of rank m. Recall that for a holomorphic vector bundle with an Hermitian metric there is a unique metric connection compatible with the holomorphic structure, i.e. a connection whose $(0,1)$-part is the canonical $\overline{\partial}$-operator. The connection form of such a canonical connection is expressed in terms of a holomorphic frame by a type $(1,0)$-form. In the associated frame bundle, which is a holomorphic principal $GL(m,\mathbb{C})$-bundle, the connection form is expressed as Maurer-Cartan form of the fiber plus the type $(1,0)$ connection form of the base, being a type $(1,0)$-form on the total space.

Returning to the general principal bundle P_G, let Θ be its curvature form of θ. An element X in the Lie algebra $\mathfrak{h}$ of H defines a G-invariant vector field on P_G. By the abuse of notation we will denote it by the same letter X. Let $I^p(G)$ be set of all G-invariant polynomials on $\mathfrak{g}$ of degree p. For any $\varphi \in I^p(G)$ with $p \geqslant m$ we define f_φ by

$$f_\varphi(X) = \int_M \varphi(\theta(X) + \Theta).$$

Then it is not difficult to see f_φ is independent of the choice of type $(1,0)$-connection θ. From this one sees that f_φ defines an element of $I^{p-m}(H)$. This is one way of expressing the equivariant cohomology.

Let us consider the special case where $G = GL(m,\mathbb{C})$ and P_G is a frame bundle of the holomorphic tangent bundle of the complex manifold M, and where H is a subgroup of the automorphism group of M. Then $I^*(G)$ is generated by the i-th elementary symmetric functions c_i of eigenvalues.

Now we introduce a larger family of integral invariants, c.f. [**15**]. This is closely related to the asymptotic Chow semistability. For $\varphi \in I^k(G)$ we put

$$\begin{aligned}\mathcal{F}_\varphi(X) &= (m-k+1)\int_M \varphi(\Theta)\wedge u_X\,\omega^{m-k} \\ &\quad + \int_M \varphi(\theta(X)+\Theta)\wedge\omega^{m-k+1},\end{aligned} \tag{3}$$

where θ is any type $(1,0)$-connection of P_G and Θ its curvature form.

THEOREM 1 ([**15**]). *$\mathcal{F}_\varphi(X)$ is independent of the choice of $\omega \in [\omega_0]$ and type $(1,0)$-connection θ.*

This family of integral invariants includes obstruction to the existence of Kähler metrics with harmonic Chern forms obtained by S. Bando [**2**]. Let M be a compact Kähler manifold and $[\omega_0]$ an arbitrary Kähler class. For any Kähler form $\omega \in [\omega_0]$ let $c_k(\omega)$ be its k-th Chern form, and $Hc_k(\omega)$ the harmonic part of $c_k(\omega)$. Then there is a $(k-1,k-1)$-form F_k such that

$$c_k(\omega) - Hc_k(\omega) = \frac{i}{2\pi}\partial\overline{\partial}F_k.$$

We define $f_k : \mathfrak{h}(M) \to \mathbb{C}$ by

$$f_k(X) = \int_M L_X F_k \wedge \omega^{m-k+1}.$$

Then f_k is independent of the choice of $\omega \in [\omega_0]$, and becomes a Lie algebra character. If there exits an $\omega \in [\omega_0]$ such that $c_k(\omega)$ is a harmonic (k,k)-form with respect to ω, then we have a Kähler form with $F_k = 0$. Thence we have $f_k = 0$. Namely f_k obstructs the existence of Kähler form in $[\omega_0]$ with harmonic k-th Chern form. In the case when $k = 1$, being a Kähler metric with harmonic first Chern form is equivalent to being Kähler metric of constant scalar curvature (this follows from the Biachi identity). Therefore f_1 obstructs the existence of Kähler metrics in $[\omega_0]$ of constant scalar curvature.

When P_G is the frame bundle of the holomorphic tangent bundle of M and θ is the Levi-Civita connection with respect to the Kähler form ω then we claim that

$$\mathcal{F}_{c_k}(X) = (m-k+1)f_k(X).$$

This is because the second term on the right hand side of (3) vanishes for $\varphi = c_k$. The first term of (3) coincides with $f_k(X)$ because $Hc_k(\omega)\wedge\omega^{m-k}$ is harmonic and coincides with the volume form $\omega^m/m!$ up to a constant multiple so that we can use equation (2). The vanishing of the second term is somewhat nontrivial. The proof can be done using the fact that $\theta(X)$ is conjugate to $L(X) = L_X - \nabla_X$ which is equal to $\nabla X = \nabla grad^{1,0}u$ for some smooth function u and then the fact that we take determinant both for fiber indices and base indices in

$$\begin{aligned}&\int_M c_p(\theta(X), \overbrace{\Theta, \cdots, \Theta}^{p-1}) \wedge \omega^{m-p+1} \\ &= \int_M c_m(\overbrace{\omega\otimes I, \cdots, \omega\otimes I}^{m-p}, \omega\otimes L(X), \overbrace{\Theta, \cdots, \Theta}^{p-1}),\end{aligned}$$

so that we get from this symmetry

$$\begin{aligned} RHS &= \int_M c_m(\omega\otimes I,\cdots,\omega\otimes I, i\partial\overline{\partial}u\otimes I,\Theta,\cdots,\Theta) \\ &= -\int_M \overline{\partial}c_m(\omega\otimes I,\cdots,\omega\otimes I, i\partial u\otimes I,\Theta,\cdots,\Theta) \\ &= 0. \end{aligned}$$

Let us now see that $\mathcal{F}_{Td^\ell}$ with $1 \leqslant \ell \leqslant m$ give obstructions to asymptotic Chow semistability where Td^ℓ denotes the ℓ-th Todd polynomial. Geometric invariant theory says that if one wishes to form a moduli space with good property such as Hausdorff property and compactifiability one has to collect only semistable ones ([**24**]). The definitions of stability and semistability are as follows. Let V be a vector space. (Typically the vector space of homogeneous polynomials of degree d with $n+1$ variables with coefficients in $\mathbb{C}$. These describe the hypersurfaces of degree d in $\mathbb{P}^n(\mathbb{C})$.) Let G be a subgroup of $SL(V)$. An element $x \in V$ is said to be stable if Gx is closed and the stabilizer of x is a finite subgroup of G. An element $x \in V$ is said to be semistable if the closure of Gx does not contain the origin $\mathbf{o}$.

Let $L \to M$ be an ample line bundle. Put $V_k := H^0(M, L^k)^*$ and let $\Phi_{|L^k|} : M \to \mathbb{P}(V_k)$ be the Kodaira embedding determined by L^k. Let d be the degree of M in $\mathbb{P}(V_k)$. A point in the product $\mathbb{P}(V_k^*) \times \cdots \times \mathbb{P}(V_k^*)$ of $m+1$ copies of $\mathbb{P}(V_k^*)$ determines $m+1$ hyperplanes $H_1, \cdots, H_{m+1}$ in $\mathbb{P}(V_k)$. The set of all $m+1$ hyperplanes $H_1, \cdots, H_{m+1}$ such that $H_1 \cap \cdots \cap H_{m+1} \cap M$ is not empty defines a divisor in $\mathbb{P}(V_k^*) \times \cdots \times \mathbb{P}(V_k^*)$. But since the degree of M is d, this divisor is defined by $\hat{M}_k \in (Sym^d(V_k))^{\otimes m+1}$. Of course $\hat{M}_k$ is defined up to a constant. The point $[\hat{M}_k] \in \mathbb{P}((Sym^d(V_k))^{\otimes m+1})$ is called the Chow point of (M, L^k). M is said to be Chow stable with respect to L^k if $\hat{M}_k$ is stable under the action of $SL(V_k)$ on $(Sym^d(V_k))^{\otimes m+1}$. M is said to be asymptotically Chow stable with respect to L if there exists a $k_0 > 0$ such that $\hat{M}_k$ is stable for all $k \geqslant k_0$. Asymptotic Chow semistability is defined similarly. The stabilizer $\hat{G}_k \subset SL(V_k)$ of $\hat{M}_k$ is a finite covering of a subgroup G_k of the automorphism group $\mathrm{Aut}(M)$ of M. If we denote by $\mathrm{Aut}(M, L)$ the subgroup of $\mathrm{Aut}(M)$ consisting of the elements which lift to an action on L, then G_k is a subgroup of $\mathrm{Aut}(M, L)$.

Suppose that a holomorphic vector field X is so chosen that it generates a one-parameter subgroup $\mathbb{C}^*$ in $\mathrm{Aut}(M, L)$, and a lifting of the $\mathbb{C}^*$-action on L is chosen. This means that a normalization of the Hamiltonian function u_X is given, with the average not being necessarily zero. Note that the lifting of the $\mathbb{C}^*$-action on L is not unique, and a different choice of the lifting gives rise to a normalization of the Hamiltonian function with an integer's difference [**16**]. Put $d_k = \dim H^0(M, L^k)$, let w_k be the weight of the $\mathbb{C}^*$-action on $\wedge^{d_k} H^0(M, L^k)$. Then by the equivariant Riemann-Roch theorem w_k is equal to the degree 1 term in t of the integral of the following (c.f. [**9**]):

$$e^{k(\omega + tu_X)} Td(tL(X) + \Theta) = \sum_{p=0}^{\infty} \frac{k^p}{p!}(\omega + tu_X)^p \sum_{q=0}^{\infty} Td^{(q)}(tL(X) + \Theta).$$

Here $Td^{(q)}$ denotes the q-th Todd polynomial and $L(X) = \nabla_X - L_X$. By expressing this degree 1 term explicitly we get

$$w_k = \sum_{p=0}^{m+1} \frac{k^p}{p!} \int_M (\omega^p \wedge Td^{(m-p+1)}(L(X)+\Theta) + p\,\omega^{p-1} \wedge u_X\, Td^{(m-p+1)}(\Theta)). \tag{4}$$

First of all, from [**17**] one sees

$$\int_M Td^{(m+1)}(L(X)+\Theta) = 0. \tag{5}$$

Hence the term $p=0$ in (4) vanishes.

Next, if (M,L) is asymptotically semistable then a suitable lift $\widetilde{X}$ of X to L induces a subgroup $\mathbb{C}^*$ in $SL(H^0(M,L^k))$ for all k at once, and moreover

$$f_{c_1^{m+1}}(\widetilde{X}) = 0, \tag{6}$$

see [**15**] for the detail. This means that for a Kähler form $\omega \in c_1(L)$ if we choose a connection θ of the associated bundle $P_{\mathbb{C}^*}$ of L so that its Chern form is equal to ω and put $u_X = \theta(\widetilde{X})$ then we have

$$\int_M u_X \omega^m = 0. \tag{7}$$

Namely (6) is equivalent to choosing the normalization of u_X so that its average is 0. Thus the term with $p = m+1$ in (4) also vanishes.

As we mentioned above the $\mathbb{C}^*$-action induces a subgroup of $SL(H^0(M,L^k))$, which means $w_k = 0$ for all k all at once. Combining (5) and (7) with this we get the following.

THEOREM 2 ([**15**]). *If (M,L) is asymptotically Chow semistable, then for $1 \leqslant \ell \leqslant m$ we have*

$$\mathcal{F}_{Td^{(\ell)}}(X) = 0. \tag{8}$$

The case $\ell = 1$ implies the vanishing of f_1.

Note in passing that under the assumption that $\mathrm{Aut}(M,L)$ is discrete, Donaldson [**8**] obtained the following results. The Kähler form of the Fubini-Study metric of $\mathbb{P}(V_k)$ is denoted by ω_{FS}.

(1) Suppose that $\mathrm{Aut}(M,L)$ is discrete and that M is asymptotically Chow stable. If the sequence of Kähler forms $\omega_k := \frac{2\pi}{k}\Phi^*_{|L^k|}(\omega_{FS})$ belonging in $c_1(L)$ converges in C^∞ to ω_∞, then ω_∞ has constant scalar curvature.
(2) Suppose that $\mathrm{Aut}(M,L)$ is discrete and that $\omega_\infty \in 2\pi c_1(L)$ has constant scalar curvature. Then M is asymptotically Chow stable with respect to L, and ω_k converges in C^∞ to ω_∞.
(3) Suppose that $\mathrm{Aut}(M,L)$ is discrete. Then a Kähler metric of constant scalar curvature in $2\pi c_1(L)$ is unique.

Donaldson quoted (and probably was inspired by) a result of Luo [**18**] who proved that the balanced condition of the projective imbedding of the polarized manifold implies the Hilbert stability, so Donaldson stated his results in terms of Hilbert stability. However the balanced condition is equivalent to Chow stability when there are no non-zero holomorphic vector fields as was indicated in Zhang [**33**] or Phong-Sturm [**26**], and moreover Chow stability implies Hilbert stability as is seen in section 4 below.

The case where $\mathrm{Aut}(M, L)$ is not discrete is treated by T. Mabuchi in [**19**], [**20**], [**21**].

3. Symplectic geometry and scalar curvature

The results of this section are due to X. Wang [**32**], but some statements and proofs are modified a bit to the author's taste.

Let (Z, Ω) be a Kähler manifold and suppose a compact Lie group K acts on Z as holomorphic isometries. Then the complexification K^c of K also acts on Z as biholomorphisms. The actions of K and K^c induce homomorphisms of the Lie algebras $\mathfrak{k}$ and $\mathfrak{k}^c$ to the real Lie algebra $\Gamma(TZ)$ of all smooth vector fields on Z, both of which we denote by ρ. If $\xi + i\eta \in \mathfrak{k}^c$ with $\xi,\ \eta \in \mathfrak{k}$, then

$$\rho(\xi + i\eta) = \rho(\xi) + J\rho(\eta),$$

where J is the complex structure of Z. Suppose $[\Omega]$ is an integral class and there is a holomorphic line bundle $L \to Z$ with $c_1(L) = [\Omega]$. There is an Hermitian metric h of L^{-1} such that its Hermitian connection θ satisfies

$$-\frac{1}{2\pi} d\theta = \Omega.$$

Suppose we have a lifting of K^c to L^{-1}, so that we have a moment map $\mu : Z \to \mathfrak{k}^*$ because the lifting of K-action to L is equivalent to defining a moment map (see [**11**], section 6.5). Let $\pi : L^{-1} \to Z$ be the projection and $\pi(p) = x$ with $p \in L^{-1} - \textit{zero section}$, $x \in Z$. Denote by $\Gamma = K^c \cdot x$ the K^c-orbit of x in Z, and $\widetilde{\Gamma} = K^c \cdot p$ be the K^c-orbit of p in L^{-1}. We say that $x \in Z$ is polystable with respect to the K^c-action if the orbit $\widetilde{\Gamma}$ is closed in L^{-1}. Consider the function $h : \widetilde{\Gamma} \to \mathbb{R}$ defined by

$$h(\gamma) = \log |\gamma|^2.$$

Fundamental facts are

- h has a critical point if and only if the moment map $\mu : Z \to \mathfrak{k}^*$ has a zero along Γ:
- h is a convex function.

For these facts refer again to [**11**], section 6.5. These imply the following two propositions.

PROPOSITION 1. *A point $x \in Z$ is polystable with respect to the action of K^c if and only if the moment map μ has a zero along Γ.*

PROPOSITION 2. *The set $\{x \in \Gamma \mid \mu(x) = 0\}$ has only one component, and the orbit $Stab(x)^c \cdot x$ of the complexification of the stabilizer at x through x is connected even if $Stab(x)^c$ is not connected.*

For a given $x \in Z$ we extend $\mu(x) : \mathfrak{k} \to \mathbb{R}$ complex linearly to $\mu(x) : \mathfrak{k}^c \to \mathbb{C}$. For notational convenience we denote by K_x (resp. $(K^c)_x$) the stabilizer of x in K (resp. K^c), and by $\mathfrak{k}_x$ and $(\mathfrak{k}^c)_x$ the Lie algebra of K_x and $(K^c)_x$. Define $f_x : (\mathfrak{k}^c)_x \to \mathbb{C}$ to be the restriction of $\mu(x) : \mathfrak{k}^c \to \mathbb{C}$ to $(\mathfrak{k}^c)_x$. Note that $(K^c)_{gx} = g(K^c)_x g^{-1}$.

PROPOSITION 3 (Wang [**32**]). *Fix $x_0 \in Z$. Then for $x \in K^c \cdot x_0$, f_x is K^c-equivariant in that $f_{gx}(Y) = f_x(Ad(g^{-1})Y)$. In particular if f_x vanishes at some $x \in K^c \cdot x_0$ it vanishes at all $x \in K^c \cdot x_0$. Moreover $f_x : (\mathfrak{k}^c)_x \to \mathbb{C}$ is a Lie algebra character.*

PROOF. The point of the first statement is that, although μ is just K-equivariant but not K^c-equivariant, μ is K^c-equivariant when restricted to the stabilizer subalgebras along the orbit. Write $Y \in (\mathfrak{k}^c)_x$ as $Y = \xi_1 + i\xi_2$. Then $\rho(\xi_1)_x + J\rho(\xi_2)_x = 0$. It is sufficient to show at x

$$(\rho(\eta_1) + J\rho(\eta_2)) < \mu, \xi_1 + i\xi_2 >=< \mu, -[\eta_1 + i\eta_2, \xi_1 + i\xi_2] > .$$

Using the K-equivariance of μ, J-invariance of ω and $\rho(\xi_1)_x + J\rho(\xi_2)_x = 0$ one sees that the left hand side is

$$\begin{aligned} LHS &= \omega(\rho(\xi_1), \rho(\eta_1)) + \omega(\rho(\xi_1), J\rho(\eta_2)) + i\omega(\rho(\xi_2), \rho(\eta_1)) + i\omega(\rho(\xi_2), J\rho(\eta_2)) \\ &= \omega(\rho(\xi_1), \rho(\eta_1)) - \omega(\rho(\xi_2), \rho(\eta_2)) + i\omega(\rho(\xi_2), \rho(\eta_1)) + i\omega(\rho(\xi_1), \rho(\eta_2)). \end{aligned}$$

On the other hand the right hand side is

$$\begin{aligned} RHS &= < \mu, -[\eta_1, \xi_1] + [\eta_2, \xi_2] - i[\eta_1, \xi_2] - i[\eta_2, \xi_1] > \\ &= \omega(\rho(\xi_1), \rho(\eta_1)) - \omega(\rho(\xi_2), \rho(\eta_2)) + i\omega(\rho(\xi_2), \rho(\eta_1)) + i\omega(\rho(\xi_1), \rho(\eta_2)). \end{aligned}$$

This completes the proof of K^c-equivariance. This in particular implies that f_x is $(K^c)_x$-invariant, and that f_x is a character of $(\mathfrak{k}^c)_x$. □

Suppose we are given a K-invariant inner product on $\mathfrak{k}$. Then we can identify $\mathfrak{k} \cong \mathfrak{k}^*$, and $\mathfrak{k}^*$ has a K-invariant inner product. Consider the function $\varphi : K^c \cdot x_0 \to \mathbb{R}$ defined by $\varphi(x) = |\mu(x)|^2$. We say that $x \in K^c \cdot x_0$ is an extremal point if x is a critical point of φ.

PROPOSITION 4 (Wang [**32**]). *Let $x \in K^c \cdot x_0$ be an extremal point. Then we have a decoposition*

$$(\mathfrak{k}^c)_x = (\mathfrak{k}_x)^c \oplus \sum_{\lambda > 0} \mathfrak{k}^c_\lambda$$

where $\mathfrak{k}^c_\lambda$ is λ-eigenspace of $\mathrm{ad}(i\mu(x))$, and $i\mu(x)$ lies in the center of $(\mathfrak{k}_x)^c$. In particular $(\mathfrak{k}_x)^c = (\mathfrak{k}^c)_x$ if and only if $\mu(x) = 0$.

PROOF. First of all if x is an extremal point and $\{x_t\}$ is a curve in $K^c \cdot x_0$ with $x(0) = x$ then

$$\begin{aligned} 0 = \frac{d}{dt}\varphi(x_t)|_{t=0} &= < \dot{x}_t\mu, \mu > (x) \\ &= \omega(\rho(\mu(x)), \dot{x}) = g(J\rho(\mu(x)), \dot{x}). \end{aligned}$$

Taking $\dot{x}(0) = \rho(i\mu(x))$ shows that $\mu(x) \in \mathfrak{k}_x$. Thus $ad(i\mu(x)) : \mathfrak{k}^c \to \mathfrak{k}^c$ preserves the subspace $(\mathfrak{k}^c)_x$. Since $\mathfrak{k}^c$ has a K- invariant Hermitian inner product, $ad(i\mu(x)) : \mathfrak{k}^c \to \mathfrak{k}^c$ and its restriction to $(\mathfrak{k}^c)_x$ are self-adjoint. Hence all the eigenvalues of $ad(i\mu(x))$ are real numbers. Let $X = \xi + i\eta \in (\mathfrak{k}^c)_x$ be an eigenvector with eigenvalue λ. We claim that

$$\lambda\|X\| = 2\|\rho(\eta)_x\|.$$

To see this let $< \cdot, \cdot >$ be the Hermitian inner product anti-$\mathbb{C}$-linear in the first factor and $\mathbb{C}$-linear in the second factor. Then

$$\begin{aligned}
\lambda < X, X > &= < ad(i\mu(x))X, X >=< ad(\mu(x)), -iX > \\
&= < \mu(x), -[\overline{X}, iX] > \\
&= ((\rho(\xi) - i\rho(\eta)) < \mu, iX >)(x) \\
&= g(iJ\rho(\xi) - J\rho(\eta), \rho(\xi) - i\rho(\eta))_x. \qquad (9)
\end{aligned}$$

But since $X \in (\mathfrak{k}^c)_x$ we have $\rho(X)_x = \rho(\xi)_x + J\rho(\eta)_x = 0$. Thus

$$\begin{aligned}
(9) &= g(i\rho(\eta) - J\rho(\eta), -J\rho(\eta) - i\rho(\eta))_x \\
&= 2g(\rho(\eta), \rho(\eta))_x.
\end{aligned}$$

This shows that $\lambda \geqslant 0$. Equality holds if and only if the real and the imaginary parts of X are in $\mathfrak{k}_x$. Thus the zero eigenspace is the complexification $(\mathfrak{k}_x)^c$ of $\mathfrak{k}_x$. This completes the proof of Proposition 3.4.

□

Let (M, ω_0, J_0) be a compact Kähler manifold with a fixed Kähler form ω_0. Denote by Z the set of all ω-compatible integral complex structures J with respect to which (M, ω_0, J) is a Kähler manifold. Then the tangent space of Z at J is the space of smooth symmetric $(2,0)$-tensors $Sym^2(T^{*1,0})$, and the natural L^2-inner product gives a Kähler structure on Z. Let $\mathfrak{k}$ be the Lie algebra consisting of all smooth functions with average 0 with respect to the measure $\omega^m/m!$ endowed with the Poisson bracket with respect to ω. Let K be the corresponding Lie group, namely the group of symplectomorphisms generated by Hamiltonian diffeomorphisms.

Theorem 3 (Fujiki [**12**], Donaldson [**7**]). *The map $\mu : Z \to \mathfrak{k}^*$ given by*

$$< \mu(J), f >= \int_M S_J f \omega^m$$

with S_J the scalar curvature of (M, ω, J) is a moment map for the action of K.

In this infinite dimensional setting there is no action of the complexification K^c. But there is an action of the complexification at the Lie algebra level. This gives a foliation in Z. The leaves consist of (ω_0, J)'s which correspond by Moser's theorem to (ω, J_0)'s with $[\omega] = [\omega_0]$.

Now we apply Proposition 3.2, 3.3 and 3.4 to Z formally. Then Proposition 3.2 gives the uniqueness of Kähler metrics of constant scalar curvature modulo the action of the automorphism group and the fact that K-energy is bounded from below and is attained on the space of Kähler forms of constant scalar curvature, since h corresponds to the K-energy (c.f. [**29**]). Detailed analysis has been carried out by [**3**], [**21, 22**], [**10**], [**5**]. Proposition 3.3 gives the character f_1 above. Proposition 3.4 gives Calabi-Lichnerowicz-Matsushima theorem ([**4**]). Thus the standard results on Kähler metrics of constant scalar curvature can be seen in the framework of stability in the infinite dimensional moment map picture. These observations are due to Wang [**32**] (and also indicated implicitly by Donaldson [**7**]).

4. K stability

In [**30**] Tian defined the notion of K-stability for Fano manifolds and proved that if a Fano manifold carries a Kähler-Einstein metric then M is weakly K-stable. Tian's K-stability considers the degenerations of M to normal varieties and uses a generalized version of the invariant f_1 defined by Ding and Tian ([**6**]). Note that this generalized invariant is only defined for normal varieties.

Further Donaldson re-defined in [**9**] the invariant f_1 for general polarized varieties (or even projective schemes) and also re-defined the notion of K-stability for (M, L). The new definition does not require M to be Fano nor the central fibers of degenerations to be normal. We now briefly review Donaldson's definition of K stability.

Let $\Lambda \to N$ be an ample line bundle over an n-dimensional projective scheme. We assume that a $\mathbb{C}^*$-action as bundle isomorphisms of Λ covering the $\mathbb{C}^*$-action on N.

For any positive integer k, there is an induced $\mathbb{C}^*$ action on $W_k = H^0(N, \Lambda^k)$. Put $d_k = \dim W_k$ and let w_k be the weight of $\mathbb{C}^*$-action on $\wedge^{d_k} W_k$. By the Riemann-Roch and the equivariant Riemann-Roch theorems d_k and w_k are polynomials in k of degree n and $n+1$ respectively. Therefore w_k/kd_k is bounded from above as k tends to infinity. For sufficiently large k we expand

$$\frac{w_k}{kd_k} = F_0 + F_1 k^{-1} + F_2 k^{-2} + \cdots .$$

For an ample line bundle L over a projective variety M, a test configuration of degree r consists of the following.
(1) A family of schemes $\pi : \mathcal{M} \to \mathbb{C}$:
(2) $\mathbb{C}^*$-action on $\mathcal{M}$ covering the usual $\mathbb{C}^*$-action on $\mathbb{C}$:
(3) $\mathbb{C}^*$-equivariant line bundle $\mathcal{L} \to \mathcal{M}$ such that
- for $t \neq 0$ one has $M_t = \pi^{-1}(t) \cong M$ and $(M_t, \mathcal{L}|_{M_t}) \cong (M.L^r)$,
- the Hilbert polynomial $\sum_{p=0}^{n}(-1)^p \dim H^p(M_t, L_t^r)$ does not depend on t, in particular for r sufficiently large $\dim H^0(M_t, L_t) = \dim H^0(M, L)$ for all $t \in \mathbb{C}$.

$\mathbb{C}^*$-action induces a $\mathbb{C}^*$-action on the central fiber $L_0 \to M_0 = \pi^{-1}(0)$. Moreover if (M, L) admits a $\mathbb{C}^*$-action, then one obtains a test configuration by taking the direct product $M \times \mathbb{C}$. This is called a product configuration.

DEFINITION 1. *(M, L) is said to be K-stable if the F_1 of the central fiber (M_0, L_0) is greater than or equal to 0 for all test configurations, and the equality occurs only if the test configuration is product.*
Conjecture([**9**]) : A Kähler metric of constant scalar curvature will exist in the Kähler class $c_1(L)$ if and only if (M, L) is K stable.

Now we come back to the property of F_1. The motivation for the Conjecture is the following lemma. Recall that Λ was an ample line bundle with $\mathbb{C}^*$-action over a projective scheme N and that F_1 was defined for (N, Λ). Suppose that N is nonsingular algebraic variety and take any Kähler form ω in $c_1(\Lambda)$. Denote by ρ and σ the Ricci form and the scalar curvature of ω respectively.

LEMMA 1 ([**9**]). *If N is a nonsingular projective variety then*

$$F_1 = \frac{-1}{2vol(N, \omega)} f_1(X)$$

where X is the infinitesimal generator of the $\mathbb{C}^*$-action and f_1 is the integral invariant defined in section 2.

PROOF. Let us denote by n the complex dimension of N. Expand $h^0(\Lambda^k)$ and $w(k)$ as

$$h^0(\Lambda^k) = a_0 k^n + a_1 k^{n-1} + \cdots ,$$
$$w(k) = b_0 k^{n+1} + b_1 k^n + \cdots .$$

Then by the Riemann-Roch and the equivariant Riemann-Roch formulae

$$a_0 = \frac{1}{n!}\int_N c_1(\Lambda)^n = vol(N),$$

$$a_1 = \frac{1}{2(n-1)!}\int_N \rho \wedge c_1(\Lambda)^{n-1} = \frac{1}{2n!}\int_N \sigma\omega^n,$$

$$b_0 = \frac{1}{(n+1)!}\int_N (n+1)u_X\omega^n,$$

$$b_1 = \frac{1}{n!}\int_N n u_X \omega^{n-1} \wedge \frac{1}{2}c_1(N) + \frac{1}{n!}\int_N \operatorname{div} X\ \omega^n.$$

The last term of the previous integral is zero because of the divergence formula. Thus

$$\frac{w(k)}{kh^0(k)} = \frac{b_0}{a_0}(1 + (\frac{b_1}{b_0} - \frac{a_1}{a_0})k^{-1} + \cdots)$$

from which we have

$$\begin{aligned}
F_1 &= \frac{b_0}{a_0}(\frac{b_1}{b_0} - \frac{a_1}{a_0}) = \frac{1}{a_0^2}(a_0 b_1 - a_1 b_0) \\
&= \frac{1}{2vol(N)}\int_N u_X(\sigma - \frac{1}{vol(N)}\int_N \sigma\frac{\omega^n}{n!})\frac{\omega^n}{n!} \\
&= \frac{1}{2vol(N)}\int_N u_X \Delta F\frac{\omega^n}{n!} \ = \ \frac{-1}{2vol(N)}\int_N XF\frac{\omega^n}{n!} \\
&= \frac{-1}{2vol(N)}f_1(X)
\end{aligned}$$

□

Let V be a vector space over $\mathbb{C}$ and ρ a one parameter subgroup of $SL(V)$. Let $[v] \in \mathbb{P}(V)$ and $\lambda \in \mathbb{C}^*$. Suppose $[\rho(\lambda)v] \to [v_0] \in \mathbb{P}(V)$ as $\lambda \to 0$. Then we have an endomorphism $\rho(\lambda) : \mathbb{C}v_0 \to \mathbb{C}v_0$. The weight of this endomorphism is called Mumford weight of (v, ρ) and is denoted by $\mu(v, \rho)$. We say that $[v] \in \mathbb{P}(V)$ is semistable (resp. stable) with respect to ρ iff $\mu(v, \rho) \leqslant 0$ (resp. $\mu(v, \rho) < 0$). We also say that $[v] \in \mathbb{P}(V)$ is polystable iff $\mu(v, \rho) < 0$ or $\rho(\mathbb{C}^*)$ is contained in $Stab(v)$. The Hilbert-Mumford criterion says that $[v] \in \mathbb{P}(V)$ is semistable (resp. polystable) with respect to a subgroup G of $SL(V)$ iff $[v] \in \mathbb{P}(V)$ is semistable (resp. polystable) with respect to arbitrary one parameter subgroup of G.

Let us define Hilbert stability of a polarized variety (M, L). Suppose L^r is a very ample line bundle with $h^i(L^r) = 0$ for $i > 0$. Then $h(r) := h^0(L^r)$ can be computed by Riemann-Roch theorem. If we fix an isomorphism $H^0(L^r) \cong \mathbb{C}^{h(r)}$ this gives an embedding $\Phi_{|L^r|} : M \to \mathbb{P}^{h(r)-1}$. A different choice of the isomorphism gives a transformation by an element of $SL(h(r))$. When k is sufficiently large we have an exact sequence

$$0 \to I_k \to S^k H^0_M(L^r) \to H^0_M(L^{kr}) \to 0,$$

where I_k denotes the set of all polynomials of degree k vanishing along the image of M. The k-th Hilbert point of (M, L^r) is the point in the Grassmannian

$$x_{k,r} \in G = G(S^k\mathbb{C}^{h(r)*}; h(rk))$$

determined by the identification $H^0_M(L^r) \cong \mathbb{C}^{h(r)}$.

We say that $(M.L)$ is Hilbert (semi)stable with respect to r iff the image of $x_{r,k} \in G$ of the Plücker embedding $G \to \mathbb{P}^{\binom{h(r)+k-1}{h(rk)}}$ is (semi)stable for all large k.

Fact (c.f. [**24**], Proposition 2.1). *Let L be a very ample line bundle with $h^i(L) = 0$ for $i > 0$, and ρ a one parameter subgroup of $SL(h^0(L))$. Let $\widetilde{w}$ be the Mumford weight of the Hilbert point $x_k \in G(S^k\mathbb{C}^{h^0(L)*}; h(k))$ with respect to ρ, and e be the Mumford weight of the Chow point of (M, L) with respect to ρ. Then we have*

$$\widetilde{w}(k) = Cek^{m+1} + O(k^m)$$

with positive constant C.

This says if $e < 0$ then $\widetilde{w}(k) < 0$ for large k, namely Chow stability implies Hilbert stability. If $\widetilde{w}(k) \leqslant 0$ for all k, then $e \leqslant 0$, namely Hilbert semistable implies Chow semistable.

Now let $\widetilde{w}(r, k)$ be the Mumford weight of $x_{r,k}$. We wish to express this in terms of $w(r)$ which was the weight for $H^0(L^r)$ of the one parameter group ρ in $SL(h^0(L))$. As ρ lies in $SL(h^0(L))$ we have to renormalize the one parameter group so that in lies in $SL(h^0(L^r))$. After this renormalization we find by putting $s = rk$

$$\begin{aligned}
\widetilde{w}(r,k) &= -w(s) + \frac{w(r)}{rh(r)}sh(s) \\
&= sh(s)\left(\frac{w(r)}{rh(r)} - \frac{w(s)}{rh(s)}\right) \\
&= sh(s)(F_1(r^{-1} - s^{-1}) + O(r^{-2} - s^{-2})).
\end{aligned}$$

Theorem 4 ([**27**], [**28**]). *If we put $\widetilde{w}(r,k) = \frac{1}{rh(r)}\sum_{i,j=0}^{m+1} a_{i,j}r^{i+j}k^j$ then*

(1) $a_{m+1,m+1} = 0$:
(2) The Chow weight e_r of (M, L^r) is given by $e_r = \frac{Cr^m}{h(r)}\sum_{i=0}^{m} a_{i,m+1}r^i$ with a positive constant C:
(3) $a_{m,m+1}$ and F_1 have the same sign.

This result says that if $e_r \leqslant 0$ for all large r then $F_1 \leqslant 0$, namely that asymptotic Chow semistability implies K-semistability.

Next we turn to the slope stability. Let Y be a closed subscheme of M. Let $\mathcal{M}$ be the blow-up of $M \times \mathbb{C}$ along $Y \times \{0\}$, and P be the exceptional divisor. Denote by $\pi : \mathcal{M} \to M \times \mathbb{C} \to M$ the composition of the projections. Consider the line bundle $\mathcal{L}_c = \pi^*L - cP$. Suppose $\mathcal{L}_c$ is ample. In fact this is the case if c is less than the Seshadri constant of Y, see Proposition 6.7, [**27**]. Consider the action $\mathbb{C}^*$-action on $M \times \mathbb{C}$ sending (x, t) to (x, zt) by $z \in \mathbb{C}^*$. This action naturally lifts to $\mathcal{L}_c$. We may consider this test configuration. It turns out that the condition of K-stability can be written in terms of Y, L and c. To express this let x be a positive rational number with $0 < x < c$ and take large k with $kx \in \mathbb{Z}$. Let $\mathcal{I}_Y^{xk}L^k$ be the sheaf of sections L vanishing along Y to order xk. Expand $h^0(\mathcal{I}_Y^{xk}L^k)$ as

$$h^0(\mathcal{I}_Y^{xk}L^k) = a_0(x)k^n + a_1(x)k^{n-1} + O(k^{n-2}).$$

Define the slope of Y by

$$\mu(\mathcal{I}_Y) = \mu(\mathcal{I}_Y, L, c) = \frac{\int_0^c (a_1(x) + \frac{a_0'(x)}{2})dx}{\int_0^c a_0(x)dx}.$$

We also put

$$\mu(M) = \frac{a_1}{a_0}.$$

This corresponds to the slope for the empty scheme. The K-stability for the above test configuration is equivalent to

$$\mu(\mathcal{I}_Y) < \mu(M).$$

(M, L) is said to be slope stable iff $\mu(\mathcal{I}_Y) < \mu(M)$ for every subscheme Y. In conclusion, the K-stability implies the slope stability. It might be possible to find a connection to the multiplier ideal sheaves invented by Nadel [**25**].

References

[1] T. Aubin : Equations du type de Monge-Ampère sur les variétés kählériennes compactes, C. R. Acad. Sci. Paris, **283**, 119-121 (1976)

[2] S. Bando : An obstruction for Chern class forms to be harmonic, preprint (1983)

[3] S. Bando and T. Mabuchi : Uniqueness of Einstein Kähler metrics modulo connected group actions, Algebraic Geometry, Sendai 1985, Adv. Stud. Pure Math., vol 10, pp.11-40, Noth-Holland, Amsterdam and Kinokuniya, Tokyo, (1987)

[4] E. Calabi : Extremal Kähler metrics II, Differential geometry and complex analysis, (I. Chavel and H.M. Farkas eds.), 95-114, Springer-Verlag, Berline-Heidelberg-New York, (1985)

[5] X.X. Chen and G. Tian : Geometry of Kähler metrics and holomorphic foliations by discs, math.DG/0409433.

[6] W. Ding and G.Tian : Kähler-Einstein metrics and the generalized Futaki invariant, Invent. Math., **110**, 315-335 (1992)

[7] S.K. Donaldson : Remarks on gauge theory, complex geometry and four-manifold topology, in 'Fields Medallists Lectures' (Atiyah, Iagolnitzer eds.), World Scientific, 1997, 384-403.

[8] S.K. Donaldson : Scalar curvature and projective embeddings, I, J. Differential Geometry, 59(2001), 479-522.

[9] S.K. Donaldson : Scalar curvature and stability of toric varieties, J. Differential Geometry, 62(2002), 289-349.

[10] S.K. Donaldson : Scalar curvature and projective embeddings, II, math.DG/0407534.

[11] S.K. Donaldson and P.B. Kronheimer : The geometry of four manifolds, Oxford Mathematical Monographs, Claren Press, Oxford, 1990.

[12] A. Fujiki : Moduli space of polarized algebraic manifolds and Kähler metrics, Sugaku Expositions, 5(1992), 173-191.

[13] A. Futaki : An obstruction to the existence of Einstein Kähler metrics, Invent. Math. **73**, 437-443 (1983)

[14] A. Futaki : On compact Kähler manifolds of constant scalar curvature, Proc. Japan Acad., Ser. A, **59**, 401-402 (1983)

[15] A. Futaki : Asymptotic Chow stability and integral invariants, to appear in Intern. J. Math.

[16] A. Futaki and T. Mabuchi : *Moment maps and symmetric multilinear forms associated with symplectic classes*, Asian J. Math., **6**, 349–372 (2002).

[17] A. Futaki and S. Morita : *Invariant polynomials on compact complex manifolds*, Proc. Japan Acad. **60**, 369–372 (1984).

[18] H. Luo : Geometric criterion for Gieseker-Mumford stability of polarized manifolds, J. Diff. Geom. 49(1998), 577-599.

[19] T. Mabuchi : An obstruction to asymptotic semistability and approximate critical metrics, Osaka J. Math., 41(2004), 463-472. math.DG/0404210.

[20] T. Mabuchi : Stability of extremal Kähler manifolds, Osaka J. Math., 41(2004), 563-582. math.DG/0404211.

[21] T. Mabuchi : An energy-theoretic approach to the Hitchin-Kobayashi correspondence for manifolds, I, to appear in Invent. Math.
[22] T. Mabuchi : An energy-theoretic approach to the Hitchin-Kobayashi correspondence for manifolds, II, preprint.
[23] Y. Matsushima : Sur la structure du groupe d'homéomorphismes d'une certaine variété kaehlérienne, Nagoya Math. J., **11**, 145-150 (1957)
[24] D. Mumford : Stability of projective varieties, L'Enseignement Mathematiques, 23(1977), 39-110.
[25] A. Nadel : Multiplier ideal sheaves and existence of Kähler-Einstein metrics of positive scalar curvature, Ann. of Math., **132**, 549-596 (1990)
[26] D.H. Phong and J. Sturm : Stability, energy functionals, and Kähler-Einstein metrics, Comm. Anal. Geom., vol. 11(2003), 563-597, arXiv: math.DG/0203254.
[27] J. Ross : Instability of polarized algebraic varieties, PhD. Thesis, Imperial College, London, 2003.
[28] J. Ross and R.P. Thomas : Slope stability of projective varieties, preprint.
[29] G. Tian : The K-energy of hypersurfaces and stability, Comm. Anal. Geom., **2**(1994), 239-265.
[30] G. Tian : Kähler-Einstein metrics with positive scalar curvature, Invent. Math., **130**, 1-37 (1997)
[31] S.-T. Yau : On Calabi's conjecture and some new results in algebraic geometry, Proc. Nat. Acad. Sci. USA, **74**, 1798-1799 (1977)
[32] X. Wang : Moment maps, Futaki invariant and stability of projective manifolds, to appear in Comm. Anal. Geom.
[33] S. Zhang : Heights and reductions of semi-stable varieties, Compositio Math. 104(1996), 77-105.

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