**DIFFERENTIAL GEOMETRY AND ITS APPLICATIONS**
Proc. Conf. Prague, August 30 – September 3, 2004
Charles University, Prague (Czech Republic), 2005, 229 – 246

# The $d\delta$–lemma for weakly Lefschetz symplectic manifolds

Marisa Fernández, Vicente Muñoz and Luis Ugarte

Abstract. For a symplectic manifold $(M, \omega)$, not necessarily hard Lefschetz, we prove a version of the Merkulov $d\delta$–lemma ([**16, 3**]). We also study the $d\delta$–lemma and related cohomologies for compact symplectic solvmanifolds.

## 1. Introduction

Let $(M, \omega)$ be a symplectic manifold, that is, $M$ is a differentiable manifold of dimension $2n$ with a closed non-degenerate 2–form $\omega$, the *symplectic form*. Denote by $\Omega^k(M)$ the space of the differential $k$–forms on $M$. According to Libermann [**11**] and Brylinski [**2**] there is a *symplectic star operator* $* : \Omega^k(M) \longrightarrow \Omega^{2n-k}(M)$ associated to the symplectic form $\omega$ satisfying $*^2 = Id$ (see Section 2 for the definition). Such an operator is the symplectic analogue of the Hodge star operator on oriented Riemannian manifolds. Then, one can define the *codifferential* $\delta = \pm * d *$ which satisfies $\delta^2 = 0$ and $d\delta + \delta d = 0$ (although $\delta$ does not satisfy a Leibniz rule [**14**]).

As in Riemannian Hodge theory, a $k$–form $\alpha \in \Omega^k(M)$ is said to be *coclosed* if $\delta\alpha = 0$, *coexact* if $\alpha = \delta\beta$ for some $\beta$, and *symplectically harmonic* if it is closed and coclosed. But, unlike the case of Riemannian manifolds, there are many symplectically harmonic forms which are exact. This is the reason for which for any $k \geqslant 0$, we define the space of *harmonic cohomology* $H^k_{\mathrm{hr}}(M, \omega)$ of degree $k$ to be the subspace of the de Rham cohomology $H^k(M)$ consisting of all cohomology classes which contain at least one symplectically harmonic $k$–form.

Mathieu [**15**] and, independently, Yan [**21**] proved that $H^k_{\mathrm{hr}}(M, \omega) = H^k(M)$ for all $k$ if and only if $(M, \omega)$ satisfies the hard Lefschetz property, i.e. the map

$$L^{n-k} \colon H^k(M) \longrightarrow H^{2n-k}(M)$$

given by $L^{n-k}[\alpha] = [\omega^{n-k} \wedge \alpha]$ is a surjection for all $k \leqslant n-1$. On the other hand, for *compact* symplectic manifolds, Merkulov and Cavalcanti ([**16, 3**]) showed that the existence of symplectic harmonic forms in every de Rham cohomology class is equivalent to the symplectic *$d\delta$–lemma*, that is, to the identities

$$\mathrm{Im}\, d \cap \ker \delta = \mathrm{Im}\, d\delta = \mathrm{Im}\, \delta \cap \ker d, \tag{1}$$

which mean that if $\alpha$ is a symplectically harmonic $k$–form and either is exact or coexact, then $\alpha = d\delta\beta$ for some $k$–form $\beta$.

Consider the subcomplex $(\Omega^*_\delta(M, \omega), d)$ of the de Rham complex $(\Omega^*(M), d)$ of $M$, where $\Omega^k_\delta(M, \omega)$ is the space of the coclosed $k$–forms. We denote by $H^*_\delta(M, \omega)$

This paper is in final form and no version of it will be published elswhere.

its cohomology and by $i$ the natural map

$$i\colon H^k_\delta(M,\omega) \longrightarrow H^k(M), \tag{2}$$

for all $k \geqslant 0$. In **[7]** Guillemin proved that if $M$ is compact, then the map $i$ is bijective if and only if $(M,\omega)$ is hard Lefschetz or, equivalently, it satisfies the $d\delta$–lemma.

In this paper, we aim to generalize these results to symplectic manifolds which are not hard Lefschetz. Recall the following definition from **[5]**.

Definition 1. *A symplectic manifold $(M,\omega)$ of dimension $2n$ is said to be $s$–*Lefschetz*, where $0 \leqslant s \leqslant n-1$, if the map*

$$L^{n-k}\colon H^k(M) \longrightarrow H^{2n-k}(M)$$

*is an epimorphism for all $k \leqslant s$. (If $M$ is compact, then we actually have that $L^{n-k}$ are isomorphisms because of Poincaré duality.)*

Whenever $(M,\omega)$ is not hard Lefschetz, there is some integer number $s \geqslant 0$ such that $(M,\omega)$ is $s$–Lefschetz, but not $(s+1)$–Lefschetz. Note that $(M,\omega)$ is $(n-1)$–Lefschetz if it satisfies the hard Lefschetz theorem.

Concerning the harmonic cohomology for such manifolds, we have the following result.

Theorem 1. **[6]** *Let $(M,\omega)$ be a symplectic manifold of dimension $2n$ and let $s \leqslant n-1$. Then the following statements are equivalent:*

(i) *$(M,\omega)$ is $s$–Lefschetz.*
(ii) *$H^k_{\mathrm{hr}}(M,\omega) = H^k(M)$ for every $k \leqslant s+2$, and $H^{2n-k}_{\mathrm{hr}}(M,\omega) = H^{2n-k}(M)$ for every $k \leqslant s$.*
(iii) *$H^{2n-k}_{\mathrm{hr}}(M,\omega) = H^{2n-k}(M)$ for every $k \leqslant s$.*

Notice that Theorem 1 implies that every de Rham cohomology class of $M$ admits a symplectically harmonic representative if and only if $(M,\omega)$ is hard Lefschetz, which is the result proved independently by Mathieu and Yan **[15, 21]**.

For any non-hard Lefschetz symplectic manifold, it seems interesting to understand how the level $s$ at which the Lefschetz property is lost affects to other properties of the manifold, such as the above mentioned $d\delta$–lemma, or to the properties of the map $i$. Our purpose in this paper is to explore these questions, as we explain below.

In Section 2 we recall some properties of the spaces of harmonic cohomology. In Section 3, we sharpen the result of Merkulov and the result of Guillemin by using the concept of $s$–Lefschetz property. We need first to weaken the condition of the $d\delta$–lemma to the following

Definition 2. *Let $(M,\omega)$ be a symplectic manifold of dimension $2n$, and $0 \leqslant s \leqslant n-1$. We say that $(M,\omega)$ satisfies the $d\delta$–*lemma up to degree *$s$ if*

$$\begin{array}{ll} \operatorname{Im} d \cap \ker \delta = \operatorname{Im} d\delta = \operatorname{Im} \delta \cap \ker d, & \text{on } \Omega^k(M), \text{ for } k \leqslant s, \\ \operatorname{Im} d \cap \ker \delta = \operatorname{Im} d\delta, & \text{on } \Omega^{s+1}(M). \end{array} \tag{3}$$

Therefore, if $(M,\omega)$ satisfies the $d\delta$–lemma up to degree $s$, and $\alpha \in \Omega^{\leqslant s}(M)$ is symplectically harmonic and either is exact or coexact then $\alpha = d\delta\beta$ for some $\beta$; moreover, if $\alpha \in \Omega^{s+1}(M)$ is symplectically harmonic and exact then $\alpha = d\delta\beta$ for some $\beta$.

Following the approach in Cavalcanti's proof **[3]** of the result of Merkulov we prove the following theorem.

THEOREM 2. *($d\delta$–lemma for weakly Lefschetz manifolds). Let $(M,\omega)$ be a compact symplectic manifold of dimension $2n$ and let $s \leqslant n-1$. Then the following statements are equivalent:*

(i) *$(M,\omega)$ is $s$–Lefschetz.*
(ii) *$(M,\omega)$ satisfies the $d\delta$–lemma up to degree $s$.*
(iii) *The identities (1) hold on $\Omega^{\geqslant(2n-s)}(M)$, and $\operatorname{Im}\delta \cap \ker d = \operatorname{Im} d\delta$ holds on $\Omega^{2n-s-1}(M)$.*

In Section 3 we also show the following theorem regarding the map (2) for weakly symplectic manifolds.

THEOREM 3. *Let $(M,\omega)$ be a compact symplectic manifold of dimension $2n$ and let $s \leqslant n-1$. Then the following statements are equivalent:*

(i) *$(M,\omega)$ is $s$–Lefschetz.*
(ii) *The map $i\colon H^k_\delta(M,\omega) \longrightarrow H^k(M)$ is bijective for all $k \leqslant s+1$ and for $k \geqslant 2n-s$.*
(iii) *The map $i\colon H^k_\delta(M,\omega) \longrightarrow H^k(M)$ is bijective for all $k \geqslant 2n-s$.*

The harmonic cohomology of compact symplectic nilmanifolds has been studied by different authors (see [**21, 9, 20**]). In Section 4 we consider compact solvmanifolds $M = \Gamma\backslash G$, where $G$ is a simply connected solvable Lie group whose Lie algebra $\mathfrak{g}$ is completely solvable, i.e., the map $\operatorname{ad}_X\colon \mathfrak{g} \longrightarrow \mathfrak{g}$ has only real eigenvalues for any $X \in \mathfrak{g}$, and $\Gamma$ is a discrete subgroup of $G$ such that the quotient $M = \Gamma\backslash G$ is compact. We show that the harmonic cohomology of $(M = \Gamma\backslash G, \omega)$ is isomorphic to the harmonic cohomology at the level of the invariant forms. We exhibit some examples of compact symplectic solvmanifolds $M$ which are $s$–Lefschetz but not $(s+1)$–Lefschetz, for small values of $s$, and so the map $i$ is bijective for $k \geqslant 2n-s$ and they satisfy the $d\delta$–lemma up to degree $s$. We detect that they do not satisfy the $d\delta$–lemma up to degree $s+1$ by exhibiting an invariant symplectically harmonic $(s+1)$–form $x$ such that $x \in \operatorname{Im}\delta$ but $x \notin \operatorname{Im} d$. We also find an invariant class $u \in H^{2n-s-1}_\delta(M,\omega)$ such that $i(u) = 0$ in $H^{2n-s-1}(M)$.

## 2. Harmonic cohomology of $s$–Lefschetz manifolds

We recall some definitions and results about the spaces of harmonic cohomology classes that we will need in the following sections. Let $(M,\omega)$ be a symplectic manifold of dimension $2n$. Denote by $\Omega^*(M)$ the algebra of differential forms on $M$, by $\mathfrak{X}(M)$ the Lie algebra of vector fields on $M$, and by $\mathcal{F}(M)$ the algebra of differentiable functions on $M$. Since $\omega$ is a non-degenerate 2–form, we have the volume form $v_M = \frac{\omega^n}{n!}$, and the isomorphism

$$\natural : \mathfrak{X}(M) \longrightarrow \Omega^1(M)$$

defined by $\natural(X) = \iota_X(\omega)$ for $X \in \mathfrak{X}(M)$, where $\iota_X$ denotes the contraction by $X$. We extend $\natural$ to an isomorphism of graded algebras $\natural : \bigoplus_{k\geqslant 0} \mathfrak{X}^k(M) \longrightarrow \bigoplus_{k\geqslant 0} \Omega^k(M)$, where $\mathfrak{X}^k(M)$ denotes the space of the skew-symmetric $k$–vectors fields. Libermann (see [**11, 12**]) defined the *symplectic star operator*

$$* : \Omega^k(M) \longrightarrow \Omega^{2n-k}(M)$$

by the condition

$$*(\alpha) = (-1)^k \iota_{\natural^{-1}(\alpha)}(v_M).$$

This operator can be also defined in terms of the skew-symmetric bivector field $G$ dual to $\omega$, that is, $G = -\natural^{-1}(\omega)$. ($G$ is the unique non-degenerate Poisson structure [**13**] associated with $\omega$.) Denote by $\Lambda^k(G)$, $k \geqslant 0$, the associated pairing $\Lambda^k(G) : \Omega^k(M) \times \Omega^k(M) \longrightarrow \mathcal{F}(M)$ which is $(-1)^k$–symmetric (i.e. symmetric for even $k$, anti-symmetric for odd $k$). Imitating the Hodge star operator for oriented Riemannian manifolds, Brylinski [**2**] defined the *symplectic star operator* by the condition $\beta \wedge (*\alpha) = \Lambda^k(G)(\beta, \alpha)v_M$, for $\alpha, \beta \in \Omega^k(M)$. An easy consequence is that $*^2 = \mathrm{d}$.

Koszul [**10**] introduced the differential $\delta \colon \Omega^k(M) \longrightarrow \Omega^{k-1}(M)$ on any Poisson manifold $M$, with Poisson tensor $G$, by the condition

$$\delta = [\iota_G, d],$$

and he proved that $\delta^2 = d\delta + \delta d = 0$. Later work by Brylinski [**2**], shows that the Koszul differential is a *symplectic codifferential* of the exterior differential with respect to the *symplectic star operator*, that is,

$$\delta\alpha = (-1)^{k+1} * d * (\alpha),$$

for $\alpha \in \Omega^k(M)$. As in Riemannian Hodge theory, a $k$–form $\alpha \in \Omega^k(M)$ is said to be *coclosed* if $\delta\alpha = 0$, *coexact* if $\alpha = \delta\beta$ for some $\beta$, and *symplectically harmonic* if it is closed and coclosed. Notice that Koszul definition of $\delta$ implies that if $\alpha$ is closed, then $\delta\alpha$ is exact. In [**14**] it is proved the following Leibniz rule for $\delta$. If $f$ is an arbitrary differentiable function on $M$ and $\alpha \in \Omega^*(M)$, then

$$\delta(f\alpha) = f\,\delta\alpha - \iota_{X_f}(\alpha),$$

where $X_f$ is the Hamiltonian vector field of $f$, i.e., $\iota_{X_f}(\omega) = df$.

Let $\Omega^k_{\mathrm{hr}}(M, \omega) = \{\alpha \in \Omega^k(M) \mid d\alpha = \delta\alpha = 0\}$ be the space of the symplectically harmonic $k$–forms. For the de Rham cohomology classes of $M$, we consider the vector space

$$H^k_{\mathrm{hr}}(M, \omega) = \frac{\Omega^k_{\mathrm{hr}}(M, \omega)}{\Omega^k_{\mathrm{hr}}(M, \omega) \cap \operatorname{Im} d},$$

consisting of the cohomology classes in $H^k(M)$ containing at least one symplectically harmonic form.

For $p, k \geqslant 0$ we define

$$L^p : \Omega^k(M) \longrightarrow \Omega^{2p+k}(M)$$

by $L^p(\alpha) = \omega^p \wedge \alpha$ for $\alpha \in \Omega^k(M)$. In [**21**] it is proved the property following

LEMMA 1. [**21**] *(Duality on differential forms). The map*

$$L^{n-k} \colon \Omega^k(M) \longrightarrow \Omega^{2n-k}(M)$$

*is an isomorphism for* $0 \leqslant k \leqslant n-1$.

Since $\omega^p$ is closed, we have

$$[L^p, d] = L^p \circ d - d \circ L^p = 0,$$

and the map $L^p$ induces a map $L^p : H^k(M) \longrightarrow H^{2p+k}(M)$ on cohomology. However, the isomorphisms of Lemma 1 do not imply special properties on the maps on cohomology (see Definition 1). Relations between the operators $\iota_G$, $L$, $d$ and $\delta$ were proved by Yan in [**21**]. Here we mention the following

$$\iota_G = - * L*, \quad [\iota_G, \delta] = 0, \quad [L, \delta] = -d,$$

which implies that if $\alpha$ is coclosed then $d\alpha$ is coexact, and if $\alpha$ is a symplectically harmonic form then $L\alpha$ and $\iota_G\alpha$ are symplectically harmonic. Also in [**21**] the following is proved.

Lemma 2. [**21**] *(Duality on harmonic forms). The map*

$$L^{n-k}\colon \Omega^k_{\mathrm{hr}}(M,\omega) \longrightarrow \Omega^{2n-k}_{\mathrm{hr}}(M,\omega)$$

*is an isomorphism for* $0 \leqslant k \leqslant n-1$.

Lemma 2 implies that the homomorphism

$$L^{n-k}\colon H^k_{\mathrm{hr}}(M,\omega) \longrightarrow H^{2n-k}_{\mathrm{hr}}(M,\omega)$$

is surjective. (Notice that the duality on harmonic forms may be not satisfied at the level of the spaces $H^*_{\mathrm{hr}}(M,\omega)$.) Since $H^{2n-k}_{\mathrm{hr}}(M,\omega)$ is a subspace of the de Rham cohomology $H^{2n-k}(M)$, we conclude that (see [**9**, Corollary 1.7])

$$H^{2n-k}_{\mathrm{hr}}(M,\omega) = \mathrm{Im}(L^{n-k}\colon H^k_{\mathrm{hr}}(M,\omega) \longrightarrow H^{2n-k}(M)).$$

A nonzero $k$–form $\alpha$, with $k \leqslant n$, is called *primitive* (or *effective*) if $L^{n-k+1}(\alpha) = 0$. Thus, any 1–form is primitive.

Lemma 3. [**12**, page 46] *If $\alpha$ is a primitive $k$–form, then there is a constant $c$ such that its symplectic star operator $*\alpha$ satisfies $*\alpha = c\,L^{n-k}(\alpha)$.*

Notice that the previous lemma implies that every closed primitive $k$–form is symplectically harmonic, and in particular $H^1(M) = H^1_{\mathrm{hr}}(M,\omega)$. For the classes in $H^2(M)$, Mathieu proved that any cohomology class of degree 2 has a symplectically harmonic representative, i.e., $H^2(M) = H^2_{\mathrm{hr}}(M,\omega)$.

Lemma 4. *Let $\alpha$ a $k$–form with $k \leqslant n$. Then, $\alpha$ is primitive if and only if $\iota_G(\alpha) = 0$.*

Proof. It follows from the identity $\iota_G = -*L*$ and Lemma 3. If $\alpha$ is primitive, $\iota_G(\alpha) = -*L*(\alpha) = -*c\,L^{n-k+1}(\alpha) = 0$. □

Lemma 5. *If $\alpha$ is a primitive $k$–form then, for all $j \leqslant n-k$, there is a non-zero constant $c_{j,k}$ such that $\iota^j_G L^j(\alpha) = c_{j,k}\,\alpha$.*

Proof. In [**21**] it is proved the relation $[\iota_G, L] = A$, where $A = \sum(n-k)\pi_k$, $\pi_k$ being the projection. Thus, for $j = 1$ we have that $\iota_G L(\alpha) = A\alpha + L(\iota_G\alpha) = (n-k)\alpha$ because $\alpha$ is primitive. Suppose that $\iota^j_G L^j(\alpha) = c_{j,k}\,\alpha$ for some $j < n-k$ with $c_{j,k}$ a non-zero constant. Hence, $\iota^{j+1}_G L^{j+1}(\alpha) = \iota^j_G\iota_G L L^j(\alpha) = \iota^j_G L\iota_G L^j(\alpha) + (n-k-2j)\iota^j_G L^j(\alpha) = \iota^j_G L\iota_G L^j(\alpha) + (n-k-2j)c_{j,k}\,(\alpha)$ by the induction hypothesis. After $p$ times we get that

$$\iota^{j+1}_G L^{j+1}(\alpha) = \iota^{j-p}_G L\iota^{p+1}_G L^j(\alpha) + (p+1)(n-k-2j+p)c_{j,k}\,\alpha.$$

Therefore, for $p = j-1$ and using the induction hypothesis we conclude that $\iota^{j+1}_G L^{j+1}(\alpha) = c_{j+1,k}\,\alpha$, with $c_{j+1,k} = (j+1)(n-k-j)c_{j,k}$ a non-zero constant. □

## 3. The $d\delta$–lemma for $s$–Lefschetz manifolds

This section is devoted to the study of the $d\delta$–lemma for symplectic manifolds which are not necessarily hard Lefschetz. By Definition 2, $(M,\omega)$ satisfies the $d\delta$-lemma up to degree $s$ if $\operatorname{Im} d \cap \ker \delta = \operatorname{Im} d\delta = \operatorname{Im} \delta \cap \ker d$ on $\Omega^k(M)$, for $k \leqslant s$ and $\operatorname{Im} d \cap \ker \delta = \operatorname{Im} d\delta$ on $\Omega^{s+1}(M)$. By applying duality with the symplectic $*$-operator, this is equivalent to

$$
\begin{aligned}
&\operatorname{Im} \delta \cap \ker d = \operatorname{Im} d\delta = \operatorname{Im} d \cap \ker \delta, && \text{on } \Omega^{2n-k}(M), \text{ for } k \leqslant s,\\
&\operatorname{Im} \delta \cap \ker d = \operatorname{Im} d\delta, && \text{on } \Omega^{2n-s-1}(M).
\end{aligned} \tag{4}
$$

Let us see the one implication (the other one is proved in an analogous way). Suppose that $(M,\omega)$ satisfies the $d\delta$-lemma up to degree $s$. If $\alpha_{2n-k} \in \Omega^{2n-k}(M)$, $0 \leqslant k \leqslant s+1$, satisfies that $\alpha_{2n-k} \in \operatorname{Im} \delta \cap \ker d$, then $*\alpha_{2n-k}$ is a $k$–form in $\operatorname{Im} d \cap \ker \delta = \operatorname{Im} d\delta$, so there is a $k$–form $\beta_k$ such that $*\alpha_{2n-k} = d\delta(\beta_k)$ and hence $\alpha_{2n-k} = *d\delta(\beta_k) = -\delta d(*\beta_k) = d\delta(*\beta_k)$. The equality $\operatorname{Im} d \cap \ker \delta = \operatorname{Im} d\delta$ on $\Omega^{\geqslant(2n-s)}(M)$ is proved analogously.

Note that if $(M,\omega)$ satisfies the $d\delta$–lemma up to degree $n-1$ then both (3) and (4) hold for $s = n-1$, and hence $(M,\omega)$ satisfies the $d\delta$–lemma since then (1) also holds on the space $\Omega^n(M)$.

In order to prove Theorems 2 and 3 we need the following results.

Lemma 6. *Let $(M,\omega)$ be a symplectic manifold of dimension $2n$, and let $\alpha$ be a $k$–form. Then*

(i) *$d\delta(L^p(\alpha)) = L^p(d\delta(\alpha))$ for all $p \geqslant 0$.*

(ii) *If $\alpha$ is primitive, then $d\delta(\alpha)$ is also primitive.*

Proof. Since $[L,\delta] = -d$, we see that $\delta L = L\delta + d$. Thus, $d\delta(L^p(\alpha)) = d(L\delta + d)L^{p-1}(\alpha) = dL\delta L^{p-1}(\alpha)$. Proceeding in this fashion $p$ times, and using that $L$ and $d$ commute, we obtain (i). Now to show (ii) we have, using (i), that $L^{n-k+1}(d\delta(\alpha)) = d\delta(L^{n-k+1}(\alpha)) = 0$ since $\alpha$ is a primitive $k$–form. □

Lemma 7. *Let $(M,\omega)$ be a symplectic manifold of dimension $2n$, let $\beta$ be a $r$–form, and let $\alpha = L^p(\beta)$, with $p \geqslant 0$. If $\delta\beta$ is exact, then $\delta\alpha$ is also exact.*

Proof. Write $\delta\beta = d\gamma$. Using $[L,\delta] = -d$, we have $\delta\alpha = \delta L^p(\beta) = (L\delta + d)L^{p-1}(\beta) = L\delta L^{p-1}(\beta) + dL^{p-1}(\beta)$. Proceeding in a similar way with the first summand, after $p$ steps, we get

$$\delta\alpha = \delta L^p(\beta) = L^p\delta(\beta) + p\, dL^{p-1}(\beta) = d(L^p(\gamma) + pL^{p-1}(\beta)),$$

which proves the lemma. □

Consider a $(2n-i)$–form $\psi$ on $(M,\omega)$ with $i \leqslant n$. According to the duality on differential forms, there is a unique $i$–form $\varphi$ such that $\psi = L^{n-i}(\varphi)$. Lepage decomposition theorem **[12]** implies that $\varphi$ may be uniquely decomposed as a sum

$$\varphi = \varphi_i + L(\varphi_{i-2}) + \cdots + L^q(\varphi_{i-2q}), \tag{5}$$

with $q \leqslant [i/2]$, where $[i/2]$ being the largest integer less than or equal to $i/2$, and where the form $\varphi_{i-2j}$ is a primitive $(i-2j)$–form, for $j = 0,\dots,q$. This implies that $\psi = L^{n-i}(\varphi)$ may be uniquely decomposed as the sum

$$\psi = L^{n-i}(\varphi) = L^{n-i}(\varphi_i) + L^{n-i+1}(\varphi_{i-2}) + \cdots + L^{n-i+q}(\varphi_{i-2q}). \tag{6}$$

LEMMA 8. *Let $(M,\omega)$ be a symplectic manifold of dimension $2n$, and let $\psi = L^{n-i}(\varphi) \in \Omega^{2n-i}(M)$ with $i \leqslant n$.*

(i) *If $d\delta(\psi) = 0$, or equivalently $d\delta(\varphi) = 0$, then all the forms $\varphi_{i-2j}$ in the decomposition* (5) *and* (6) *satisfy $d\delta(\varphi_{i-2j}) = 0$.*

(ii) *If $\delta\varphi_{i-2j}$ is exact for all $j = 0, \dots, q$, then both $\delta\varphi$ and $\delta\psi$ are exact.*

PROOF. Suppose that $d\delta(\psi) = 0$. Applying $d\delta$ to (6), using Lemma 6 and the uniqueness of the decomposition, we have that

$$L^{n-i+j} d\delta(\varphi_{i-2j}) = 0,$$

for $j = 0, \dots, q$. We see that $L^{n-i+j} d\delta(\varphi_{i-2j}) = 0$ implies $d\delta(\varphi_{i-2j}) = 0$. In fact, the map $L^{n-i+2j} \colon \Omega^{i-2j}(M) \longrightarrow \Omega^{2n-i+2j}(M)$ is an isomorphism for all $j = 0, \dots, q$. So, the map $L^{n-i+j} \colon \Omega^{i-2j}(M) \longrightarrow \Omega^{2n-i}(M)$ is injective for $j = 1, \dots, q$, and it is an isomorphism for $j = 0$. Using again Lemma 6 and the duality on differential forms, one can check that $d\delta(\varphi) = 0$ implies the same result. Part (ii) follows from Lemma 7 and using that $\delta$ is a linear map. $\square$

PROPOSITION 1. *Let $(M,\omega)$ be an $s$–Lefschetz compact symplectic manifold of dimension $2n$. Then,*

$$\operatorname{Im}\delta \cap \ker d = \operatorname{Im} d \cap \operatorname{Im}\delta,$$

*on the spaces $\Omega^{\leqslant s}(M)$ and $\Omega^{\geqslant(2n-s-2)}(M)$; and*

$$\operatorname{Im} d \cap \ker\delta = \operatorname{Im} d \cap \operatorname{Im}\delta,$$

*on $\Omega^{\leqslant(s+2)}(M)$ and $\Omega^{\geqslant(2n-s)}(M)$.*

PROOF. We prove only the first identity because the second is analogous by duality using the symplectic $*$-operator. The result can be restated in the following way: if $\varphi$ is a $k$–form, with $k \leqslant s+1$ or $k \geqslant 2n-s-1$, and such that $d\delta(\varphi) = 0$, then $\delta\varphi$ is exact.

First, we show such a result for any primitive $k$–form $\varphi$ with $k \leqslant s+1$. We define the $(k-1)$–form $\gamma$ by

$$L^{n-k+1}(\gamma) = dL^{n-k}(\varphi). \tag{7}$$

Thus $\gamma$ is primitive since $L^{n-k+2}(\gamma) = dL^{n-k+1}(\varphi) = 0$. Applying $\iota_G^{n-k+1}$ in (7), using Lemma 5 and $\delta = [\iota_G, d]$, we have

$$c_{n-k+1,k-1}\gamma = \iota_G^{n-k+1} dL^{n-k}\varphi = \iota_G^{n-k}(d\iota_G + \delta)L^{n-k}\varphi.$$

Proceeding in this fashion, after $(n-k+1)$ times, we have

$$c_{n-k+1,k-1}\gamma = (d\iota_G^{n-k+1} - (n-k+1)\delta\iota_G^{n-k})L^{n-k}\varphi.$$

Since $\varphi$ is primitive, $(d\iota_G^{n-k+1})L^{n-k}\varphi = d\iota_G(c_{n-k,k}\varphi) = 0$ by Lemma 4. So, there is a non-zero constant $c$ such that $\gamma = c\,\delta\varphi$. Applying $L^{n-k+1}$ to both sides and using (7) we obtain

$$c\,L^{n-k+1}\delta\varphi = L^{n-k+1}\gamma = d(L^{n-k}\varphi).$$

By hypothesis $\delta\varphi$ is closed. Moreover the map $L^{n-k+1} \colon H^{k-1}(M) \longrightarrow H^{2n-k+1}(M)$ is an isomorphism for $k-1 \leqslant s$ since $(M,\omega)$ is compact and $s$–Lefschetz. Thus $\delta\varphi$ is exact because $L^{n-k+1}\delta\varphi$ defines the zero class.

Now we pass to the case where $\varphi$ is an arbitrary $k$–form with $k \leqslant s+1$ such that $d\delta(\varphi) = 0$. From Lemma 8 we know that every primitive form $\varphi_{i-2j}$ in the decomposition (5) satisfies $d\delta(\varphi_{i-2j}) = 0$, and so $\delta(\varphi_{i-2j})$ is exact. Now Lemma 8 implies that $\delta\varphi$ is exact.

Finally, if $\psi$ is a $k$–form with $k \geqslant 2n - s - 1$ and such that $d\delta(\psi) = 0$, then the forms $\varphi_{i-2j}$ in the decomposition (6) are of degree $\leqslant s + 1$, and they satisfy $d\delta(\varphi_{i-2j}) = 0$ by Lemma 8. Taking account the previous result for primitive forms, we conclude that all the forms $\delta\varphi_{i-2j}$ are exact, and hence $\delta\psi$ is exact by Lemma 8. □

Proposition 2. *Let $(M, \omega)$ be an $s$–Lefschetz compact symplectic manifold of dimension $2n$. We have*

(i) $\operatorname{Im}\delta \cap \ker d = \operatorname{Im} d \cap \ker \delta$ *on* $\Omega^{\leqslant s}(M)$ *and* $\Omega^{\geqslant(2n-s)}(M)$.

(ii) $(M, \omega)$ *satisfies the* $d\delta$*–lemma up to degree* $s$.

Proof. Part (i) follows directly from Proposition 1. To show (ii), we shall first prove that $\operatorname{Im}\delta \cap \ker d = \operatorname{Im} d\delta$ on the spaces $\Omega^{\geqslant(2n-s-1)}(M)$. We will prove this by induction on $s$. For $s = 0$, assume $\alpha \in \Omega^{2n}(M)$ such that $\alpha \in \operatorname{Im}\delta \cap \ker d = \operatorname{Im} d \cap \ker\delta$. Then $\alpha = 0$ because $\operatorname{Im}\delta = 0$, and so $\alpha = 0 = d\delta 0$. Now we see that $\operatorname{Im}\delta \cap \ker d = \operatorname{Im} d\delta$ on $\Omega^{2n-1}(M)$. Let $\beta = \delta\alpha$ be a $(2n-1)$–form, $\alpha \in \Omega^{2n}(M)$, such that $d\delta\alpha = 0$. Since $d\alpha = 0$ and $H^{2n}(M) = H^{2n}_{\mathrm{hr}}(M, \omega)$, there is $\tilde{\alpha} \in \Omega^{2n}(M)$ such that $\delta\tilde{\alpha} = 0$, $d\tilde{\alpha} = 0$ and $\alpha = \tilde{\alpha} + d\gamma$, for some $\gamma \in \Omega^{2n-1}(M)$. Therefore, $\beta = \delta\alpha = \delta d\gamma = d\delta(-\gamma)$.

Now take $s > 0$, and assume that if $(M, \omega)$ is $(s-1)$–Lefschetz, then $\operatorname{Im}\delta \cap \ker d = \operatorname{Im} d\delta$ on $\Omega^{\geqslant(2n-s)}(M)$. We need to prove that if $(M, \omega)$ is $s$–Lefschetz, then $\operatorname{Im}\delta \cap \ker d = \operatorname{Im} d\delta$ on $\Omega^{2n-s-1}(M)$. We will use subscripts to keep track of the spaces that the forms belong to, i.e. $\alpha_k \in \Omega^k(M)$. We consider a $(2n-s-1)$–form $\alpha_{2n-s-1}$ such that $\alpha_{2n-s-1} = \delta\alpha_{2n-s} \in \operatorname{Im}\delta \cap \ker d$. Then,

$$0 = d\alpha_{2n-s-1} = d\delta\alpha_{2n-s} = -\delta d\alpha_{2n-s},$$

which implies that $d\alpha_{2n-s}$ is a $(2n-s+1)$–form such that $d\alpha_{2n-s} \in \operatorname{Im} d \cap \ker\delta = \operatorname{Im}\delta \cap \ker d = \operatorname{Im} d\delta$ by (i) and induction hypothesis. Thus

$$d\alpha_{2n-s} = d\delta\alpha_{2n-s+1},$$

for some $\alpha_{2n-s+1} \in \Omega^{2n-s+1}(M)$, and consequently

$$d(\alpha_{2n-s} - \delta\alpha_{2n-s+1}) = 0,$$

which means that $\alpha_{2n-s} - \delta\alpha_{2n-s+1}$ defines a de Rham cohomology class in $H^{2n-s} = H^{2n-s}_{\mathrm{hr}}(M, \omega)$, the last equality by Theorem 1. Thus, there exist a symplectically harmonic $(2n-s)$–form $\beta_{2n-s}$ and $\eta_{2n-s-1} \in \Omega^{2n-s-1}(M)$ such that

$$\alpha_{2n-s} - \delta\alpha_{2n-s+1} - \beta_{2n-s} = d\eta_{2n-s-1}.$$

Applying $\delta$ to both sides we have

$$\alpha_{2n-s-1} = \delta\alpha_{2n-s} = \delta d\eta_{2n-s-1} = -d\delta\eta_{2n-s-1} \in \operatorname{Im} d\delta.$$

To end the proof, we use the duality by the symplectic $*$-operator to show that $\operatorname{Im} d \cap \ker\delta = \operatorname{Im} d\delta$ on the spaces $\Omega^{\leqslant(s+1)}(M)$. In fact, let us consider $\alpha_r$ a differential $r$–form, with $r \leqslant s + 1$, such that $\alpha_r \in \operatorname{Im} d \cap \ker\delta$. Then, $*\alpha_r$ is a $(2n-r)$–form, $2n - r \geqslant 2n - s - 1$, such that $*\alpha_r \in \operatorname{Im}\delta \cap \ker d = \operatorname{Im} d\delta$, and so $\alpha_r \in \operatorname{Im} d\delta$. The equality $\operatorname{Im} d \cap \ker\delta = \operatorname{Im}\delta \cap \ker d$ on the spaces $\Omega^{\leqslant s}(M)$ follows from (i), and this completes the proof of the $d\delta$-lemma up to degree $s$ for $(M, \omega)$. □

**Proof of Theorem 2 :** Clearly (i) implies (ii) by Proposition 2. Also (ii) implies (iii) by duality of the symplectic $*$-operator.

Let us show that (iii) implies (i). By Theorem 1, it is enough to prove that every de Rham cohomology class of degree $k$ has a symplectically harmonic representative for $2n-s \leqslant k \leqslant 2n$. Let us consider $[\gamma] \in H^k(M)$ with $2n-s \leqslant k \leqslant 2n$. Then $d\gamma = 0$, and $\delta\gamma$ is a $(k-1)$–form such that $d\delta\gamma = 0$ since $d$ and $\delta$ anticommute. This means that $\delta\gamma$ lives in $\operatorname{Im}\delta \cap \ker d$ which is equal to $\operatorname{Im} d\delta$ on forms of degree $k-1 \geqslant 2n-s-1$ by the hypothesis (iii). This implies that there is a $(k-1)$–form $\theta$ such that $\delta\gamma = d\delta\theta$. So $\delta(\gamma + d\theta) = 0$. Then, the form $\gamma + d\theta$ is symplectically harmonic and cohomologous to $\gamma$.

Remark 1. *Notice that if $(M,\omega)$ is a compact symplectic manifold of dimension $2n$ and it is $(n-2)$–Lefschetz, then the identities* (1) *hold on $\Omega^{\leqslant(n-2)}(M)$ and $\Omega^{\geqslant(n+2)}(M)$, and also $\operatorname{Im}\delta \cap \ker d = \operatorname{Im} d \cap \operatorname{Im}\delta = \operatorname{Im} d \cap \ker\delta$ on $\Omega^n(M)$, by Proposition 1. Nonetheless, if $(M,\omega)$ is not hard Lefschetz, then this last space is in general different from* $\operatorname{Im} d\delta$.

Let $\Omega^k_\delta(M,\omega) = \{\alpha \in \Omega^k(M) \mid \delta\alpha = 0\}$ be the space of the coclosed $k$–forms. Since $d$ and $\delta$ anti-commute, then $d(\Omega^k_\delta(M,\omega)) \subset \Omega^{k+1}_\delta(M,\omega)$, and so $(\Omega^*_\delta(M,\omega), d)$ is a subcomplex of the de Rham complex $(\Omega^*(M), d)$. We denote by $H^*_\delta(M,\omega)$ its cohomology, that is

$$H^k_\delta(M,\omega) = \frac{\ker(d\colon \Omega^k_\delta(M,\omega) \longrightarrow \Omega^{k+1}_\delta(M,\omega))}{\operatorname{Im}(d\colon \Omega^{k-1}_\delta(M,\omega) \longrightarrow \Omega^k_\delta(M,\omega))}.$$

Therefore, any cohomology class on $H^k_\delta(M,\omega)$ is symplectically harmonic, and we have a natural map $i_1\colon H^k_\delta(M,\omega) \longrightarrow H^k_{\mathrm{hr}}(M,\omega)$ which is always surjective but may be non-injective. The next theorem gives a necessary and sufficient condition for the injectivity of the map $i_1$. (Notice that $\Omega^*_\delta(M,\omega) = \bigoplus \Omega^k_\delta(M,\omega)$ is a vector space but not an algebra because the codifferential $\delta$ does not satisfy a Leibniz rule.) It is clear that there is a natural map

$$i\colon H^k_\delta(M,\omega) \longrightarrow H^k(M),$$

for all $k$. In fact, denote by $i_2$ the natural inclusion

$$i_2\colon H^k_{\mathrm{hr}}(M,\omega) \longrightarrow H^k(M).$$

Then, $i = i_2 \circ i_1$.

**Proof of Theorem 3 :** Suppose that $(M,\omega)$ is $s$–Lefschetz. By Theorem 1, $H^k_{\mathrm{hr}}(M,\omega) = H^k(M)$ for $k \leqslant s+2$ and $k \geqslant 2n-s$. Then, to show (ii) it is enough to prove that the map $i = i_1\colon H^k_\delta(M,\omega) \longrightarrow H^k_{\mathrm{hr}}(M,\omega)$ is injective for $k \leqslant s+1$ and $k \geqslant 2n-s$ because such a map is always surjective. Consider $[\alpha] \in H^k_\delta(M,\omega)$ and suppose that $[\alpha] = i[\alpha]$ defines the zero class on $H^k_{\mathrm{hr}}(M,\omega)$. Then $\alpha$ is exact, i.e. $\alpha = d\beta$ for some $\beta \in \Omega^{k-1}(M)$. But if $k \leqslant s+1$ or $k \geqslant 2n-s$, Theorem 2 implies $\alpha = d\delta\eta$ for some $\eta \in \Omega^k(M)$. Hence $\alpha = d\delta\eta \in \operatorname{Im}(d\colon \Omega^{k-1}_\delta(M,\omega) \longrightarrow \Omega^k_\delta(M,\omega))$. This means that $\alpha$ defines the zero class on $H^k_\delta(M,\omega)$, which proves (ii).

Clearly (ii) implies (iii). We show that (iii) implies (i). In fact, if $[\alpha] \in H^k_\delta(M,\omega)$, $[\alpha]$ is a harmonic cohomology class. Thus, if the map $i\colon H^k_\delta(M,\omega) \longrightarrow H^k(M)$ is bijective for $k \geqslant 2n-s$ then $H^k_{\mathrm{hr}}(M,\omega) = H^k(M)$ for $k \geqslant 2n-s$, i.e. $(M,\omega)$ is $s$–Lefschetz according to Theorem 1.

## 4. Harmonic cohomology of compact completely solvmanifolds

Let $\mathfrak{g}$ be a Lie algebra of dimension $2n$, and denote by $d$ the Chevalley-Eilenberg differential of $\mathfrak{g}$. An element $\omega \in \bigwedge^2(\mathfrak{g}^*)$ such that $d\omega = 0$ and $\omega^n \neq 0$ will be called a *symplectic form* on $\mathfrak{g}$.

Symplectic Hodge theory can be introduced for a symplectic form $\omega$ on a Lie algebra $\mathfrak{g}$ in a similar way as in Section 2. Let us define the star operator $*\colon \bigwedge^k(\mathfrak{g}^*) \longrightarrow \bigwedge^{2n-k}(\mathfrak{g}^*)$ by

$$*\,\alpha = (-1)^k \iota_{\natural^{-1}(\alpha)} \frac{\omega^n}{n!},$$

for any $\alpha \in \bigwedge^k(\mathfrak{g}^*)$, where $\natural$ denotes the isomorphism between $\bigwedge^k(\mathfrak{g})$ and $\bigwedge^k(\mathfrak{g}^*)$ extended from the natural isomorphism $\natural\colon \mathfrak{g} \longrightarrow \mathfrak{g}^*$ given by $\natural(X)(Y) = \omega(X,Y)$, for $X, Y \in \mathfrak{g}$.

We define the codifferential $\delta\colon \bigwedge^k(\mathfrak{g}^*) \longrightarrow \bigwedge^{k-1}(\mathfrak{g}^*)$ by

$$\delta\alpha = (-1)^{k+1} * d * \alpha,$$

for any $\alpha \in \bigwedge^k(\mathfrak{g}^*)$. Now, let $\bigwedge^k_{\mathrm{hr}}(\mathfrak{g}^*,\omega) = \{\alpha \in \bigwedge^k(\mathfrak{g}^*) \mid d\alpha = \delta\alpha = 0\}$, and consider the space

$$H^k_{\mathrm{hr}}(\mathfrak{g},\omega) = \frac{\bigwedge^k_{\mathrm{hr}}(\mathfrak{g}^*,\omega)}{\bigwedge^k_{\mathrm{hr}}(\mathfrak{g}^*,\omega) \cap \operatorname{Im} d}\,.$$

Then, $H^k_{\mathrm{hr}}(\mathfrak{g},\omega)$ consists of all the classes in the Chevalley-Eilenberg cohomology $H^k(\mathfrak{g})$ of $\mathfrak{g}$ containing at least one representative which is both closed and $\omega$-coclosed.

Let $G \in \bigwedge^2(\mathfrak{g})$ be given by $G = -\natural^{-1}(\omega)$. In order to study the spaces $H^k_{\mathrm{hr}}(\mathfrak{g},\omega)$ we consider the linear maps $L\colon \bigwedge^*(\mathfrak{g}^*) \longrightarrow \bigwedge^{*+2}(\mathfrak{g}^*)$, $\iota_G\colon \bigwedge^*(\mathfrak{g}^*) \longrightarrow \bigwedge^{*-2}(\mathfrak{g}^*)$ and $A\colon \bigwedge^*(\mathfrak{g}^*) \longrightarrow \bigwedge^*(\mathfrak{g}^*)$ as usual: $L\alpha$ is the wedge product by $\omega$, $\iota_G\alpha$ the contraction by $G$ and $A = \sum(n-k)\pi_k$, where $\pi_k$ is the projection onto $\bigwedge^k(\mathfrak{g}^*)$. Following [**21**], although the arguments in this special case are more direct, the following relations hold:

$$[L,\delta] = -d, \quad [\iota_G, d] = \delta, \quad [L,d] = [\iota_G,\delta] = 0,$$

and

$$[\iota_G, L] = A, \quad [A,\iota_G] = 2\,\iota_G, \quad [A,L] = -2\,L.$$

Since the standard basis $\left\{X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\ Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},\ H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right\}$ of $\mathfrak{sl}(2,\mathbb{C})$ satisfies

$$[X,Y] = H, \quad [H,X] = 2X, \quad [H,Y] = -2Y,$$

we have representations

$$\rho_1\colon \mathfrak{sl}(2,\mathbb{C}) \longrightarrow \mathfrak{gl}(\bigwedge^*(\mathfrak{g}^*)\otimes\mathbb{C}) \quad \text{and} \quad \rho_2\colon \mathfrak{sl}(2,\mathbb{C}) \longrightarrow \mathfrak{gl}(\bigwedge^*_{\mathrm{hr}}(\mathfrak{g}^*,\omega)\otimes\mathbb{C})$$

of the Lie algebra $\mathfrak{sl}(2,\mathbb{C})$ on the complex vector spaces $\bigwedge^*(\mathfrak{g}^*)\otimes\mathbb{C}$ and $\bigwedge^*_{\mathrm{hr}}(\mathfrak{g}^*,\omega)\otimes\mathbb{C}$, respectively, defined by the correspondence

$$\rho_i(X) = \iota_G, \quad \rho_i(Y) = L, \quad \rho_i(H) = A \quad (i = 1,2),$$

where $\iota_G$, $L$ and $A$ are understood for $\rho_1$ as the extension of the maps $\iota_G$, $L$ and $A$ above to the complexification $\bigwedge^*(\mathfrak{g}^*)\otimes\mathbb{C}$ of $\bigwedge^*(\mathfrak{g}^*)$, and for $\rho_2$ as the restriction of them to the subspace $\bigwedge^*_{\mathrm{hr}}(\mathfrak{g}^*,\omega)\otimes\mathbb{C}$. Notice that we can consider the restriction $\rho_2$

of the $\mathfrak{sl}(2,\mathbb{C})$ representation $\rho_1$ since if $\alpha$ is symplectically harmonic then $L\alpha$ and $\iota_G\alpha$ are symplectically harmonic.

It is well-known (see for example [**19**]) that for any representation $\rho$ of $\mathfrak{sl}(2,\mathbb{C})$ on a finite dimensional complex vector space $V$, all the eigenvalues of $\rho(H)\colon V \longrightarrow V$ are integer numbers and, if $V_k$ denotes the eigenspace of $\rho(H)$ with respect to the eigenvalue $k$, then

$$\rho(X)^k\colon V_{-k} \longrightarrow V_k \quad \text{and} \quad \rho(Y)^k\colon V_k \longrightarrow V_{-k}$$

are isomorphisms. Therefore, since $\bigwedge^r(\mathfrak{g}^*)\otimes\mathbb{C}$ and $\bigwedge^r_{\mathrm{hr}}(\mathfrak{g}^*,\omega)\otimes\mathbb{C}$ are the eigenspaces of $\rho_1(H)$ and $\rho_2(H)$, respectively, with respect to the eigenvalue $n-r$, we conclude that

$$L^k\colon \bigwedge{}^{n-k}(\mathfrak{g}^*) \longrightarrow \bigwedge{}^{n+k}(\mathfrak{g}^*)$$

and

$$L^k\colon \bigwedge{}^{n-k}_{\mathrm{hr}}(\mathfrak{g}^*,\omega) \longrightarrow \bigwedge{}^{n+k}_{\mathrm{hr}}(\mathfrak{g}^*,\omega)$$

are isomorphisms for $k \geqslant 0$.

Remark 2. *Lemmas 1 and 2 expressing duality of forms and of harmonic forms, respectively, are derived by Yan* [**21**] *from the theory of a special type of infinite dimensional representations of* $\mathfrak{sl}(2,\mathbb{C})$ *called of finite $H$-spectrum. Any finite dimensional representation of* $\mathfrak{sl}(2,\mathbb{C})$ *is of this type.*

The following result is a direct consequence of the isomorphisms $L^k$ given above. In the proof we follow the lines of [**20**, Lemma 4.3] and [**9**, Corollary 2.4], where a similar result is given for the harmonic cohomology $H^*_{\mathrm{hr}}(M)$ of a symplectic manifold $M$.

Lemma 9. *Let $\mathfrak{g}$ be a $2n$-dimensional Lie algebra with a symplectic form $\omega$. For every $k \geqslant 0$, we have*

$$H^{n-k}_{\mathrm{hr}}(\mathfrak{g},\omega) = P_{n-k}(\mathfrak{g},\omega) + L(H^{n-k-2}_{\mathrm{hr}}(\mathfrak{g},\omega)) \quad \textit{and} \quad H^{n+k}_{\mathrm{hr}}(\mathfrak{g},\omega) = L^k(H^{n-k}_{\mathrm{hr}}(\mathfrak{g},\omega)),$$

*where $P_r(\mathfrak{g},\omega) = \{[\alpha] \in H^r(\mathfrak{g}) \mid L^{n-r+1}[\alpha] = 0\}$ is the space of primitive cohomology classes of degree $r$, and $L$ denotes the product by $[\omega] \in H^2(\mathfrak{g})$.*

Proof. Let $[\alpha] \in H^{n-k}_{\mathrm{hr}}(\mathfrak{g},\omega)$. Since $L^{k+2}(\bigwedge^{n-k-2}_{\mathrm{hr}}(\mathfrak{g}^*,\omega)) = \bigwedge^{n+k+2}_{\mathrm{hr}}(\mathfrak{g}^*,\omega)$, there exists $\beta$ such that $d\beta = \delta\beta = 0$ and $L^{k+2}\beta = L^{k+1}\alpha$. Therefore, $L^{k+1}([\alpha] - L[\beta]) = 0$. Since $[\alpha] = ([\alpha] - L[\beta]) + L[\beta]$, the inclusion $H^{n-k}_{\mathrm{hr}}(\mathfrak{g},\omega) \subset P_{n-k}(\mathfrak{g},\omega) + L(H^{n-k-2}_{\mathrm{hr}}(\mathfrak{g},\omega))$ holds.

To prove the other inclusion it suffices to show that any class $[\alpha] \in P_{n-k}(\mathfrak{g},\omega)$ contains a representative $\tilde{\alpha}$ such that $\delta\tilde{\alpha} = 0$. Since $L^{k+1}[\alpha] = 0$, there exists $\gamma \in \bigwedge^{n+k+1}(\mathfrak{g}^*)$ such that $L^{k+1}\alpha = d\gamma$, so $\gamma = L^{k+1}\beta$ for some $\beta \in \bigwedge^{n-k-1}(\mathfrak{g}^*)$. Let $\tilde{\alpha} = \alpha - d\beta$. Since $L^{k+1}\tilde{\alpha} = 0$ we have that $*\tilde{\alpha}$ is proportional to $L^k\tilde{\alpha}$, therefore $\tilde{\alpha}$ is a representative of $[\alpha]$ satisfying $\delta\tilde{\alpha} = [\iota_G, d]\tilde{\alpha} = 0$.

Finally, if $[\alpha] \in H^{n+k}_{\mathrm{hr}}(\mathfrak{g},\omega)$ then there is $\beta \in \bigwedge^{n-k}_{\mathrm{hr}}(\mathfrak{g}^*,\omega)$ such that $\alpha = L^k\beta$, so $H^{n+k}_{\mathrm{hr}}(\mathfrak{g},\omega) = L^k(H^{n-k}_{\mathrm{hr}}(\mathfrak{g},\omega))$. □

Suppose that a simply connected Lie group $G$ has a discrete subgroup $\Gamma$ such that the quotient $M = \Gamma\backslash G$ is compact. Let us denote by $\mathfrak{g}$ the Lie algebra of $G$. Since any element in $\bigwedge^k(\mathfrak{g}^*)$ is identified to a left invariant form on $G$, it descends to the quotient $M$ and there is a natural injection $\bigwedge^*(\mathfrak{g}^*) \hookrightarrow \Omega^*(M)$ which commutes with the differentials.

On the other hand, if the Lie algebra $\mathfrak{g}$ of $G$ possesses a symplectic form $\omega$ then it descends to a symplectic form on $M$, which we shall also denote by $\omega$. In this case the natural injection $\bigwedge^*(\mathfrak{g}^*) \hookrightarrow \Omega^*(M)$ also commutes with the symplectic stars, and so with the $\delta$'s. Therefore, we have a natural homomorphism $H^*_{\mathrm{hr}}(\mathfrak{g}, \omega) \longrightarrow H^*_{\mathrm{hr}}(M)$.

PROPOSITION 3. *If the natural inclusion $\bigwedge^*(\mathfrak{g}^*) \hookrightarrow \Omega^*(M)$ induces an isomorphism $H^*(\mathfrak{g}) \cong H^*(M)$ in cohomology, then the inclusion $\bigwedge^*_{\mathrm{hr}}(\mathfrak{g}^*, \omega) \hookrightarrow \Omega^*_{\mathrm{hr}}(M)$ also induces an isomorphism $H^*_{\mathrm{hr}}(\mathfrak{g}, \omega) \cong H^*_{\mathrm{hr}}(M)$.*

PROOF. Since the natural homomorphism $H^k(\mathfrak{g}) \longrightarrow H^k(M)$ commutes with $L$ and it is an isomorphism, for each $k \leqslant n$ we have an isomorphism between $P_k(\mathfrak{g}, \omega)$ and the space $P_k(M) = \{[\alpha] \in H^k(M) \mid L^{n-k+1}[\alpha] = 0\}$ of primitive cohomology classes of degree $k$. Since $H^{n+k}_{\mathrm{hr}}(M) = L^k(H^{n-k}_{\mathrm{hr}}(M))$, from Lemma 9 it suffices to prove that $H^{n-k}_{\mathrm{hr}}(\mathfrak{g}, \omega) \cong H^{n-k}_{\mathrm{hr}}(M)$. But this follows easily by an inductive argument, taking into account Lemma 9 and the fact [**20**] that $H^{n-k}_{\mathrm{hr}}(M) = P_{n-k}(M) + L(H^{n-k-2}_{\mathrm{hr}}(M))$. Notice that for starting the induction, i.e. for $n-k = 0, 1, 2$, we have $H^{n-k}_{\mathrm{hr}}(M) = H^{n-k}(M) \cong H^{n-k}(\mathfrak{g}) = H^{n-k}_{\mathrm{hr}}(\mathfrak{g}, \omega)$. $\square$

Let $M = \Gamma\backslash G$ be a compact solvmanifold, that is, a compact quotient of a simply connected solvable Lie group $G$ by a discrete subgroup $\Gamma$. Suppose in addition that the Lie algebra $\mathfrak{g}$ of $G$ is completely solvable, i.e. $\mathrm{ad}_X\colon \mathfrak{g} \longrightarrow \mathfrak{g}$ has only real eigenvalues for any $X \in \mathfrak{g}$. By Hattori theorem [**8**], which is a generalization to the completely solvable context of Nomizu theorem [**17**] for nilmanifolds, the natural inclusion $\bigwedge^*(\mathfrak{g}^*) \hookrightarrow \Omega^*(M)$ induces an isomorphism $H^*(\mathfrak{g}) \cong H^*(M)$.

Let $\omega$ be a symplectic form on $M = \Gamma\backslash G$. From the results above it is clear that the harmonic cohomology only depends on the cohomology class $[\omega]$ of the symplectic form. Since $H^2(M) \cong H^2(\mathfrak{g})$ we can suppose without loss of generality that $\omega$ is invariant, that is, it stems from a symplectic form on the Lie algebra $\mathfrak{g}$.

COROLLARY 1. *Let $M = \Gamma\backslash G$ be a compact solvmanifold endowed with a symplectic form $\omega$. If the Lie algebra $\mathfrak{g}$ of $G$ is completely solvable, then the natural injection $\bigwedge^*_{\mathrm{hr}}(\mathfrak{g}^*, \omega) \hookrightarrow \Omega^*_{\mathrm{hr}}(M)$ induces an isomorphism $H^*_{\mathrm{hr}}(\mathfrak{g}, \omega) \cong H^*_{\mathrm{hr}}(M)$.*

In particular, the result holds for symplectic nilmanifolds, which has been already obtained in [**20**].

From Theorems 2 and 3 and their corresponding analogues for a Lie algebra endowed with a symplectic form, we have the following results.

COROLLARY 2. *Let $M = \Gamma\backslash G$ be a compact solvmanifold endowed with a symplectic form $\omega$. Then, the $d\delta$-lemma holds on $M$ up to degree $s$ if and only if it holds on $\mathfrak{g}$ up to degree $s$, i.e.,*

$$\begin{aligned} &d(\textstyle\bigwedge^{k-1}(\mathfrak{g}^*)) \cap \ker\delta = \delta(\textstyle\bigwedge^{k+1}(\mathfrak{g}^*)) \cap \ker d = d\delta(\textstyle\bigwedge^{k}(\mathfrak{g}^*)), \qquad \text{for } k \leqslant s, \\ &d(\textstyle\bigwedge^{s}(\mathfrak{g}^*)) \cap \ker\delta = d\delta(\textstyle\bigwedge^{s+1}(\mathfrak{g}^*)). \end{aligned}$$

COROLLARY 3. *Let $M = \Gamma\backslash G$ be a compact solvmanifold endowed with a symplectic form $\omega$. Then, the map $i$ given in (2) is bijective for all $k \geqslant 2n - s$ if and only if the map $i\colon H^k_\delta(\mathfrak{g}, \omega) \longrightarrow H^k(\mathfrak{g})$ is bijective for all $k \geqslant 2n - s$, where $H^k_\delta(\mathfrak{g}, \omega)$ denotes the cohomology of $(\bigwedge^*_\delta(\mathfrak{g}^*, \omega) = \{\alpha \in \bigwedge^*(\mathfrak{g}^*) \mid \delta\alpha = 0\},\ d)$.*

Notice that Theorems 2 and 3 imply that if two symplectic forms $\omega$ and $\omega'$ are cohomologous then the $d\delta$-lemma holds up to degree $s$ for $\omega$ if and only if it does for

$\omega'$, and the map $i$ given in (2) is bijective for all $k \geqslant 2n-s$ for $\omega$ if and only if it is so for $\omega'$. Therefore, we can consider symplectic forms up to cohomology class.

Next we consider an arbitrary symplectic form on some examples of compact completely solvable manifolds, where we show explicit calculations.

EXAMPLE. *The Kodaira-Thurston manifold* [**18**]. Let $G$ be the connected nilpotent Lie group of dimension 4 given by $G = H \times \mathbb{R}$, where $H$ is the Heisenberg group, that is, the Lie group consisting of matrices of the form

$$g = \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix},$$

where $x, y, z \in \mathbb{R}$. If $\Gamma'$ denotes the discrete subgroup of $H$ consisting of matrices whose entries $x$, $y$ and $z$ are integer numbers, then the Kodaira-Thurston manifold is the quotient space $KT = \Gamma\backslash G$, where $\Gamma = \Gamma' \times \mathbb{Z}$, is a compact manifold.

A global system of coordinates $(x, y, z)$ for $H$ is given by $x(g) = x$, $y(g) = y$, $z(g) = z$, and a standard calculation shows that a basis for the left invariant 1–forms on $H$ consists of $\{dx, dy, dz - xdy\}$. Thus, if $t$ denotes the standard coordinate for $\mathbb{R}$, then $\{\alpha = -dx, \beta = dy, \gamma = dt, \tau = dz - xdy\}$ is a basis of the dual $\mathfrak{g}^*$ of the Lie algebra $\mathfrak{g}$ of $G$ with Chevalley-Eilenberg differential given by

$$d\alpha = d\beta = d\gamma = 0, \quad d\tau = \alpha \wedge \beta.$$

So, the Chevalley-Eilenberg cohomology of $\mathfrak{g}$ is given by

$$\begin{aligned} H^0(\mathfrak{g}) &= \langle 1 \rangle, \\ H^1(\mathfrak{g}) &= \langle [\alpha], [\beta], [\gamma] \rangle, \\ H^2(\mathfrak{g}) &= \langle [\alpha\wedge\gamma], [\alpha\wedge\tau], [\beta\wedge\gamma], [\beta\wedge\tau] \rangle, \\ H^3(\mathfrak{g}) &= \langle [\alpha\wedge\beta\wedge\tau], [\alpha\wedge\gamma\wedge\tau], [\beta\wedge\gamma\wedge\tau] \rangle, \\ H^4(\mathfrak{g}) &= \langle [\alpha\wedge\beta\wedge\gamma\wedge\tau] \rangle. \end{aligned}$$

For any element $\omega \in \bigwedge^2(\mathfrak{g}^*)$ satisfying $d\omega = 0$ there exists $a, b, c, e \in \mathbb{R}$ such that

$$[\omega] = a\,[\alpha\wedge\gamma] + b\,[\beta\wedge\gamma] + c\,[\alpha\wedge\tau] + e\,[\beta\wedge\tau].$$

Since $[\omega]^2 = 2(bc - ae)[\alpha\wedge\beta\wedge\gamma\wedge\tau]$, we conclude that $[\omega]^2 \neq 0$ if and only if $ae \neq bc$. Hence, up to cohomology class, we can consider that any symplectic form on $\mathfrak{g}$ is given by

$$\omega = a\,\alpha\wedge\gamma + b\,\beta\wedge\gamma + c\,\alpha\wedge\tau + e\,\beta\wedge\tau, \qquad ae - bc \neq 0. \tag{8}$$

Moreover, notice that for the new basis of $\mathfrak{g}^*$ given by

$$\alpha' = (ae-bc)(a\,\alpha + b\,\beta), \quad \beta' = \frac{1}{ae-bc}(c\,\alpha + e\,\beta), \quad \gamma' = \frac{1}{ae-bc}\,\gamma, \quad \tau' = (ae-bc)\tau,$$

the differential $d$ expressed again as

$$d\alpha' = d\beta' = d\gamma' = 0, \quad d\tau' = \alpha' \wedge \beta'.$$

Now, with respect to this basis the symplectic form (8) is given by

$$\omega = \alpha'\wedge\gamma' + \beta'\wedge\tau'.$$

Therefore, we can suppose without loss of generality that $a = e = 1$ and $b = c = 0$ in (8).

Observe that $[\omega] \cup [\beta] = 0$ in $H^3(\mathfrak{g})$, and $\dim H^3_{\mathrm{hr}}(\mathfrak{g}, \omega) = 2 < 3 = \dim H^3(\mathfrak{g})$. It follows from Corollary 1 that for any symplectic form $\omega$ on $KT$, the compact symplectic manifold $(KT, \omega)$ is not 1–Lefschetz, and $H^k_{\mathrm{hr}}(KT, \omega) = H^k(KT)$ for $k \neq 3$, but $\dim H^3_{\mathrm{hr}}(KT, \omega) = 2 < 3 = b_3(KT)$. Notice that any non-toral compact symplectic nilmanifold $(M = \Gamma\backslash G, \omega)$ is 0–Lefschetz but not 1–Lefschetz [**1**].

We study next the $d\delta$-lemma for any symplectic form $\omega$ on the Kodaira-Thurston manifold. By Corollary 2, the $d\delta$-lemma is satisfied up to degree $s = 0$ if and only if it is satisfied at the level of the Lie algebra $\mathfrak{g}$. Let us denote by $\{X, Y, Z, T\}$ the basis of $\mathfrak{g}$ dual to $\{\alpha, \beta, \gamma, \tau\}$, and let $\omega$ be a symplectic form on $\mathfrak{g}$ given by (8) with $a = e = 1$ and $b = c = 0$. Then, the isomorphism $\natural\colon \mathfrak{g} \longrightarrow \mathfrak{g}^*$ is given by

$$\natural(X) = \gamma, \quad \natural(Y) = \tau, \quad \natural(Z) = -\alpha, \quad \natural(T) = -\beta.$$

Therefore,

$$G = -\natural^{-1}(\omega) = -X \wedge Z - Y \wedge T.$$

In degree 1, we must determine the spaces $\delta(\bigwedge^2(\mathfrak{g}^*)) \cap \ker d$, $d(\bigwedge^0(\mathfrak{g}^*)) \cap \ker \delta$ and $d\delta(\bigwedge^1(\mathfrak{g}^*))$. Notice that $\delta(\bigwedge^1(\mathfrak{g}^*)) \subset \bigwedge^0(\mathfrak{g}^*) = \mathbb{R}$ and $d(\bigwedge^0(\mathfrak{g}^*)) = \{0\}$. Using that $\delta\mu = i_G(d\mu)$ for any $\mu \in \bigwedge^2(\mathfrak{g}^*)$, an easy calculation shows that $\delta(\bigwedge^2(\mathfrak{g}^*)) = \langle\beta\rangle$, in fact $\beta = \delta(-\gamma \wedge \tau)$. Since $\ker d = \langle \alpha, \beta, \gamma\rangle$, we have

$$\delta(\bigwedge{}^2(\mathfrak{g}^*)) \cap \ker d = \langle\beta\rangle \neq \{0\} = d\delta(\bigwedge{}^1(\mathfrak{g}^*)),$$

and the $d\delta$-lemma is not satisfied in degree 1. Moreover, $d(\bigwedge^1(\mathfrak{g}^*)) \cap \ker \delta = \langle \alpha \wedge \beta\rangle \neq \{0\} = d\delta(\bigwedge^2(\mathfrak{g}^*))$.

Applying the symplectic star operator, we get that the element $\alpha \wedge \beta \wedge \gamma = *\beta \in d(\bigwedge^2(\mathfrak{g}^*)) \cap \ker \delta$, but it does not belong to the space $d\delta(\bigwedge^3(\mathfrak{g}^*)) = \{0\}$.

Therefore, for any symplectic form $\omega$ on $KT$ the $d\delta$–lemma is satisfied only up to degree 0, according to Theorem 2.

Notice that in general for any symplectic form on a nilpotent Lie algebra the map $L^{n-1}$ is never injective [**1**]. From Theorem 2 it follows that $\operatorname{Im} d \cap \ker \delta = \operatorname{Im} d\delta$ on $\bigwedge^1(\mathfrak{g}^*)$ and $\operatorname{Im} \delta \cap \ker d = \operatorname{Im} d\delta$ on $\bigwedge^{2n-1}(\mathfrak{g}^*)$, in fact these spaces are all zero, but either $\operatorname{Im} \delta \cap \ker d = \operatorname{Im} d\delta$ fails on $\bigwedge^1(\mathfrak{g}^*)$ or $\operatorname{Im} d \cap \ker \delta = \operatorname{Im} d\delta$ fails on $\bigwedge^2(\mathfrak{g}^*)$. By duality, $\operatorname{Im} d \cap \ker \delta = \operatorname{Im} d\delta$ fails on $\bigwedge^{2n-1}(\mathfrak{g}^*)$, or $\operatorname{Im} \delta \cap \ker d = \operatorname{Im} d\delta$ fails on $\bigwedge^{2n-2}(\mathfrak{g}^*)$.

Finally, we study the cohomology $H^*_\delta$. At the level of $\mathfrak{g}$, the cohomology groups $H^k_\delta(\mathfrak{g}, \omega)$ are:

$$\begin{aligned}
H^0_\delta(\mathfrak{g}, \omega) &= \langle 1\rangle, \\
H^1_\delta(\mathfrak{g}, \omega) &= \langle [\alpha], [\beta], [\gamma]\rangle, \\
H^2_\delta(\mathfrak{g}, \omega) &= \langle [\alpha \wedge \gamma], [\alpha \wedge \tau], [\beta \wedge \gamma], [\beta \wedge \tau]\rangle, \\
H^3_\delta(\mathfrak{g}, \omega) &= \langle [\alpha \wedge \beta \wedge \gamma], [\alpha \wedge \gamma \wedge \tau], [\beta \wedge \gamma \wedge \tau]\rangle, \\
H^4_\delta(\mathfrak{g}, \omega) &= \langle [\alpha \wedge \beta \wedge \gamma \wedge \tau]\rangle.
\end{aligned}$$

Therefore, $i\colon H^k_\delta(\mathfrak{g}, \omega) \longrightarrow H^k(\mathfrak{g})$ is bijective for all $k \neq 3$, because $i([\alpha \wedge \beta \wedge \gamma]) = 0$ in $H^3(\mathfrak{g})$, in fact $\alpha \wedge \beta \wedge \gamma = d(-\gamma \wedge \tau)$. From Corollary 3 we have that for any symplectic form $\omega$ on $KT$ the map $i\colon H^k_\delta(KT, \omega) \longrightarrow H^k(KT)$ is bijective for $k = 4$, but not for $k = 3$, according to Theorem 3.

EXAMPLE. *A six-dimensional solvmanifold.* Let $G$ be the connected completely solvable Lie group of dimension 6 consisting of matrices of the form

$$g = \begin{pmatrix} e^t & 0 & xe^t & 0 & 0 & y_1 \\ 0 & e^{-t} & 0 & xe^{-t} & 0 & y_2 \\ 0 & 0 & e^t & 0 & 0 & z_1 \\ 0 & 0 & 0 & e^{-t} & 0 & z_2 \\ 0 & 0 & 0 & 0 & 1 & t \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

where $t, x, y_i, z_i \in \mathbb{R}$ $(i = 1, 2)$. The Lie group $G$ has a discrete subgroup $\Gamma$ such that the quotient space $M = \Gamma\backslash G$ is compact [**4**].

A global system of coordinates $(t, x, y_1, y_2, z_1, z_2)$ for $G$ is defined by $t(g) = t$, $x(g) = x$, $y_i(g) = y_i$, $z_i(g) = z_i$, and a standard calculation shows that a basis for the left invariant 1–forms on $G$ consists of

$$\{\alpha = dt,\ \beta = dx,\ \gamma_1 = e^{-t}dy_1 - xe^{-t}dz_1,\ \gamma_2 = e^t dy_2 - xe^t dz_2,\ \tau_1 = e^{-t}dz_1,\ \tau_2 = e^t dz_2\}.$$

Hence, $\{\alpha, \beta, \gamma_1, \gamma_2, \tau_1, \tau_2\}$ is a basis of the dual $\mathfrak{g}^*$ of the Lie algebra $\mathfrak{g}$ of $G$ with Chevalley-Eilenberg differential given by

$$d\alpha = d\beta = 0, \quad d\gamma_1 = -\alpha\wedge\gamma_1 - \beta\wedge\tau_1, \quad d\gamma_2 = \alpha\wedge\gamma_2 - \beta\wedge\tau_2, \quad d\tau_1 = -\alpha\wedge\tau_1,$$
$$d\tau_2 = \alpha\wedge\tau_2.$$

Now, a direct calculation shows that the Chevalley-Eilenberg cohomology of $\mathfrak{g}$ is given by

$$\begin{aligned}
H^0(\mathfrak{g}) &= \langle 1\rangle,\\
H^1(\mathfrak{g}) &= \langle [\alpha], [\beta]\rangle,\\
H^2(\mathfrak{g}) &= \langle [\alpha\wedge\beta], [\gamma_1\wedge\tau_2 + \gamma_2\wedge\tau_1], [\tau_1\wedge\tau_2]\rangle,\\
H^3(\mathfrak{g}) &= \langle [\alpha\wedge(\gamma_1\wedge\tau_2 + \gamma_2\wedge\tau_1)], [\alpha\wedge\tau_1\wedge\tau_2], [\beta\wedge\gamma_1\wedge\gamma_2],\\
&\qquad [\beta\wedge(\gamma_1\wedge\tau_2 + \gamma_2\wedge\tau_1)]\rangle,\\
H^4(\mathfrak{g}) &= \langle [\alpha\wedge\beta\wedge\gamma_1\wedge\gamma_2], [\alpha\wedge\beta\wedge(\gamma_1\wedge\tau_2 + \gamma_2\wedge\tau_1)], [\gamma_1\wedge\gamma_2\wedge\tau_1\wedge\tau_2]\rangle,\\
H^5(\mathfrak{g}) &= \langle [\alpha\wedge\gamma_1\wedge\gamma_2\wedge\tau_1\wedge\tau_2], [\beta\wedge\gamma_1\wedge\gamma_2\wedge\tau_1\wedge\tau_2]\rangle,\\
H^6(\mathfrak{g}) &= \langle [\alpha\wedge\beta\wedge\gamma_1\wedge\gamma_2\wedge\tau_1\wedge\tau_2]\rangle.
\end{aligned}$$

For any element $\omega \in \bigwedge^2(\mathfrak{g}^*)$ satisfying $d\omega = 0$ there exists $a, b, c \in \mathbb{R}$ such that

$$[\omega] = a\,[\alpha\wedge\beta] + b\,[\gamma_1\wedge\tau_2 + \gamma_2\wedge\tau_1] + c\,[\tau_1\wedge\tau_2].$$

Since $[\omega]^3 = 6ab^2[\alpha\wedge\beta\wedge\gamma_1\wedge\gamma_2\wedge\tau_1\wedge\tau_2]$, we conclude that $[\omega]^3 \neq 0$ if and only if $ab \neq 0$.

Thus, up to cohomology class, we can consider that any symplectic form on $\mathfrak{g}$ is given by

$$\omega = a\,\alpha\wedge\beta + b\,\gamma_1\wedge\tau_2 + b\,\gamma_2\wedge\tau_1 + c\,\tau_1\wedge\tau_2, \qquad a, b \neq 0. \tag{9}$$

Let us consider the new basis of $\mathfrak{g}^*$ given by

$$\alpha' = \alpha,\ \beta' = a\,\beta,\ \gamma_1' = \sqrt{\frac{a}{b}}\,\left(b\gamma_1 + \frac{c}{2}\tau_1\right),\ \gamma_2' = \sqrt{\frac{a}{b}}\,\left(b\gamma_2 - \frac{c}{2}\tau_2\right), \tau_1' = \sqrt{\frac{b}{a}}\,\tau_1,$$
$$\tau_2' = \sqrt{\frac{b}{a}}\,\tau_2,$$

if $ab > 0$, or

$$\alpha' = -\alpha,\ \beta' = -a\,\beta,\ \gamma_1' = \sqrt{-\frac{a}{b}}\,\left(b\gamma_2 - \frac{c}{2}\tau_2\right),$$
$$\gamma_2' = \sqrt{-\frac{a}{b}}\,\left(b\gamma_1 + \frac{c}{2}\tau_1\right),\ \tau_1' = \sqrt{-\frac{b}{a}}\tau_2,\ \tau_2' = \sqrt{-\frac{b}{a}}\tau_1,$$

if $ab < 0$. The differential $d$ also expressed as

$$d\alpha' = d\beta' = 0,\quad d\gamma_1' = -\alpha'\wedge\gamma_1' - \beta'\wedge\tau_1',\quad d\gamma_2' = \alpha'\wedge\gamma_2' - \beta'\wedge\tau_2',$$
$$d\tau_1' = -\alpha'\wedge\tau_1',\quad d\tau_2' = \alpha'\wedge\tau_2'.$$

With respect to this basis the symplectic form (9) is given by

$$\omega = \alpha'\wedge\beta' + \gamma_1'\wedge\tau_2' + \gamma_2'\wedge\tau_1',$$

so we can suppose without loss of generality that $a = b = 1$ and $c = 0$ in (9).

Observe that $[\omega]\cup[\tau_1\wedge\tau_2] = 0$ in $H^4(\mathfrak{g})$, but a simple computation shows that the product by $[\omega]^2$ is an isomorphism between $H^1(\mathfrak{g})$ and $H^5(\mathfrak{g})$. Moreover, $\dim H^4_{\mathrm{hr}}(\mathfrak{g},\omega) = 2 < 3 = \dim H^4(\mathfrak{g})$. Therefore, for any symplectic form $\omega$ on $M$, the compact symplectic manifold $(M,\omega)$ is 1–Lefschetz, but not 2–Lefschetz, and Corollary 1 implies that $\dim H^k_{\mathrm{hr}}(M,\omega) = b_k(M)$ for $k \neq 4$, but $\dim H^4_{\mathrm{hr}}(M,\omega) = 2 < 3 = b_4(M)$.

Next we study the $d\delta$-lemma for any symplectic form on the compact solvmanifold $M$. Corollary 2 implies that the $d\delta$-lemma is satisfied up to degree 1 on $M$ if and only if it is satisfied on $\mathfrak{g}$. Let us denote by $\{X, Y, Z_1, Z_2, T_1, T_2\}$ the basis of $\mathfrak{g}$ dual to $\{\alpha, \beta, \gamma_1, \gamma_2, \tau_1, \tau_2\}$, and let $\omega$ be a symplectic form on $\mathfrak{g}$ given by (9) with $a = 1$, $b = 1$ and $c = 0$. Then, the isomorphism $\natural\colon \mathfrak{g} \longrightarrow \mathfrak{g}^*$ is given by

$$\natural(X) = \beta,\quad \natural(Y) = -\alpha,\quad \natural(Z_1) = \tau_2,\quad \natural(Z_2) = \tau_1,\quad \natural(T_1) = -\gamma_2,\quad \natural(T_2) = -\gamma_1.$$

Therefore, $G = -\natural^{-1}(\omega)$ is given by

$$G = -X\wedge Y - Z_1\wedge T_2 - Z_2\wedge T_1.$$

In degree 1 we must consider the spaces $\delta(\bigwedge^2(\mathfrak{g}^*))\cap\ker d$, $d(\bigwedge^0(\mathfrak{g}^*))\cap\ker\delta$ and $d\delta(\bigwedge^1(\mathfrak{g}^*))$. Since $\delta(\bigwedge^1(\mathfrak{g}^*)) \subset \bigwedge^0(\mathfrak{g}^*) = \mathbb{R}$, the $d\delta$-lemma is satisfied in degree 1 if and onlyif

$$\delta(\bigwedge{}^2(\mathfrak{g}^*))\cap\ker d = \{0\}.$$

Using that $\delta\mu = i_G(d\mu)$ for any $\mu \in \bigwedge^2(\mathfrak{g}^*)$, a direct calculation shows that the space $\delta(\bigwedge^2(\mathfrak{g}^*))$ is generated by $\gamma_1$, $\gamma_2$, $\tau_1$ and $\tau_2$. Since $\ker d = \langle\alpha, \beta\rangle$, the $d\delta$-lemma holds in degree 1.

In degree 2 we must compare the spaces $\delta(\bigwedge^3(\mathfrak{g}^*))\cap\ker d$, $d(\bigwedge^1(\mathfrak{g}^*))\cap\ker\delta$ and $d\delta(\bigwedge^2(\mathfrak{g}^*))$. It is easy to check that $d(\bigwedge^1(\mathfrak{g}^*)) \subset \ker\{\delta\colon \bigwedge^2(\mathfrak{g}^*) \longrightarrow \bigwedge^1(\mathfrak{g}^*)\}$. Therefore,

$$d(\bigwedge{}^1(\mathfrak{g}^*))\cap\ker\delta = d(\bigwedge{}^1(\mathfrak{g}^*)) = \langle d\gamma_1, d\gamma_2, d\tau_1, d\tau_2\rangle = d\delta(\bigwedge{}^2(\mathfrak{g}^*)),$$

so this space is generated by $\alpha\wedge\gamma_1 + \beta\wedge\tau_1$, $\alpha\wedge\gamma_2 - \beta\wedge\tau_2$, $\alpha\wedge\tau_1$ and $\alpha\wedge\tau_2$.

However, a long but direct calculation shows that

$$\delta(\bigwedge{}^3(\mathfrak{g}^*))\cap\ker d = \langle d\gamma_1, d\gamma_2, d\tau_1, d\tau_2, \tau_1\wedge\tau_2\rangle \ \not\subset\ d(\bigwedge{}^1(\mathfrak{g}^*))\cap\ker\delta.$$

In fact, notice that

$$\begin{aligned}\delta(\alpha\wedge\gamma_1\wedge\tau_2) &= i_G d(\alpha\wedge\gamma_1\wedge\tau_2) - d i_G(\alpha\wedge\gamma_1\wedge\tau_2)\\ &= i_G(\alpha\wedge\beta\wedge\tau_1\wedge\tau_2) + d(\alpha)\\ &= -\tau_1\wedge\tau_2,\end{aligned}$$

and $d(\tau_1\wedge\tau_2)=0$. Thus, the $d\delta$-lemma is satisfied up to degree 1, but it does not hold up to degree 2. Therefore, for any symplectic form $\omega$ on $M$ the $d\delta$–lemma is satisfied only up to degree 1, according to Theorem 2.

Notice that the element $\alpha\wedge\beta\wedge\tau_1\wedge\tau_2 = *(-\tau_1\wedge\tau_2)\in d(\bigwedge^3(\mathfrak{g}^*))\cap\ker\delta$ does not belong to the space $d\delta(\bigwedge^4(\mathfrak{g}^*))$.

Finally, the cohomology groups $H^k_\delta(\mathfrak{g},\omega)$ are given by:

$$\begin{aligned}
H^0_\delta(\mathfrak{g},\omega) &= \langle 1\rangle,\\
H^1_\delta(\mathfrak{g},\omega) &= \langle [\alpha],[\beta]\rangle,\\
H^2_\delta(\mathfrak{g},\omega) &= \langle [\alpha\wedge\beta],[\gamma_1\wedge\tau_2+\gamma_2\wedge\tau_1],[\tau_1\wedge\tau_2]\rangle,\\
H^3_\delta(\mathfrak{g},\omega) &= \langle [\alpha\wedge(\gamma_1\wedge\tau_2+\gamma_2\wedge\tau_1)],[\alpha\wedge\tau_1\wedge\tau_2],[\beta\wedge\gamma_1\wedge\gamma_2],\\
&\qquad [\beta\wedge(\gamma_1\wedge\tau_2+\gamma_2\wedge\tau_1)]\rangle,\\
H^4_\delta(\mathfrak{g},\omega) &= \langle [\alpha\wedge\beta\wedge(\gamma_1\wedge\tau_2+\gamma_2\wedge\tau_1)],[\alpha\wedge\beta\wedge\tau_1\wedge\tau_2],[\gamma_1\wedge\gamma_2\wedge\tau_1\wedge\tau_2]\rangle,\\
H^5_\delta(\mathfrak{g},\omega) &= \langle [\alpha\wedge\gamma_1\wedge\gamma_2\wedge\tau_1\wedge\tau_2],[\beta\wedge\gamma_1\wedge\gamma_2\wedge\tau_1\wedge\tau_2]\rangle,\\
H^6_\delta(\mathfrak{g},\omega) &= \langle [\alpha\wedge\beta\wedge\gamma_1\wedge\gamma_2\wedge\tau_1\wedge\tau_2]\rangle.
\end{aligned}$$

Thus, the map $i\colon H^k_\delta(\mathfrak{g},\omega)\longrightarrow H^k(\mathfrak{g})$ is bijective for all $k\neq 4$. In fact, since $d(\alpha\wedge\gamma_1\wedge\tau_2)=\alpha\wedge\beta\wedge\tau_1\wedge\tau_2$ we have that $i([\alpha\wedge\beta\wedge\tau_1\wedge\tau_2])=0$ in $H^4(\mathfrak{g})$. By Corollary 3 we conclude that for any symplectic form $\omega$ on $M$ the map $i\colon H^k_\delta(M,\omega)\longrightarrow H^k(M)$ is bijective for $k=5,6$, but not for $k=4$, according to Theorem 3.

**Acknowledgements.** We thank to G. Cavalcanti for useful conversations. This work has been partially supported through grants MCyT (Spain) Project BFM2001-3778-C03-02/03, UPV 00127.310-E-15909/2004 and MTM2004-07090-C03-01.

## References

[1] C. Benson and C.S. Gordon, Kähler and symplectic structures on nilmanifolds, *Topology* **27** (1988), 513–518.

[2] J.L. Brylinski, A differential complex for Poisson manifolds, *J. Diff. Geom.* **28** (1988), 93–114.

[3] G.R. Cavalcanti, New aspects of the $dd^c$-lemma, Ph. D. Thesis, University of Oxford, 2004.

[4] M. Fernández, M. de León and M. Saralegui, A six dimensional compact symplectic solvmanifold without Kähler structures, *Osaka J. Math.* **33** (1996), 19–35.

[5] M. Fernández and V. Muñoz, Formality of Donaldson submanifolds, Math. Zeit. **250** (2005), 149–175.

[6] M. Fernández, V. Muñoz and L. Ugarte, Weakly Lefschetz symplectic manifolds, Trans. Amer. Math. Soc., To appear.

[7] V. Guillemin, Symplectic Hodge theory and the $d\delta$–lemma, Preprint, Massachusets Institute of Technology, 2001.

[8] A. Hattori, Spectral sequence in the de Rham cohomology of fibre bundles, *J. Fac. Sci. Univ. Tokyo* **8** (1960), 298–331.

[9] R. Ibáñez, Y. Rudyak, A. Tralle and L. Ugarte, On symplectically harmonic forms on 6–dimensional nilmanifolds, *Comment. Math. Helv.* **76** (2001), 89–109.

[10] J.L. Koszul, Crochet de Schouten-Nijenhuis et cohomologie, in *Elie Cartan et les Math. d'Aujour d'Hui*, Astérisque hors-série (1985), 251–271.
[11] P. Libermann, Sur le problème d'equivalence de certaines structures infinitesimales regulieres, *Ann. Mat. Pura Appl.* **36** (1954), 27–120.
[12] P. Libermann and C. Marle, *Symplectic Geometry and Analytical Mechanics*, Kluwer, Dordrecht, 1987.
[13] A. Lichnerowicz, Les variétés de Poisson et les algébres de Lie associées, *J. Diff. Geom.* **12** (1977), 253–300.
[14] Y. Lin and R. Sjamaar, Equivariant symplectic Hodge theory and the $d_G\delta$–lemma, J. Symplectic Geom. **2** (2004), 267–278.
[15] O. Mathieu, Harmonic cohomology classes of symplectic manifolds, *Comment. Math. Helv.* **70** (1995), 1–9.
[16] S. Merkulov, Formality of canonical symplectic complexes and Frobenius manifolds, *Internat. Math. Res. Notices* **14** (1998), 723–733.
[17] K. Nomizu, On the cohomology of compact homogeneous spaces of nilpotent Lie groups, *Ann. of Math.* **59** (1954), 531-538.
[18] W.P. Thurston, Some simple examples of symplectic manifolds, *Proc. Amer. Math. Soc.* **55** (1976), 467–468.
[19] R. Wells, Differential analysis on complex manifolds. Second edition. Graduate Texts in Mathematics **65**. Springer-Verlag, New York-Berlin, 1980.
[20] T. Yamada, Harmonic cohomology groups of compact symplectic nilmanifolds, *Osaka J. Math.* **39** (2002), 363–381.
[21] D. Yan, Hodge structure on symplectic manifolds, *Adv. Math.* **120** (1996), 143–154.

M. Fernández:
Departamento de Matemáticas,
Facultad de Ciencia y Tecnología, Universidad del País
Vasco, Apartado 644, 48080 Bilbao, Spain

e-mail: `mtpferol@lg.ehu.es`

V. Muñoz:
Instituto de Matemáticas y Física Fundamental,
Consejo Superior de Investigaciones Científicas,
Serrano 113 bis, 28006 Madrid, Spain

e-mail: `vicente.munoz@imaff.cfmac.csic.es`

L. Ugarte:
Departamento de Matemáticas,
Facultad de Ciencias, Universidad de Zaragoza,
Campus Plaza San Francisco,
50009 Zaragoza, Spain

e-mail: `ugarte@unizar.es`