DIFFERENTIAL GEOMETRY AND ITS APPLICATIONS
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On the Equivariant Variational Bicomplex

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ABSTRACT. We show how the equivariant cohomology of the jet bundle is related to the cohomology of the quotient space of sections modulo the action of a group, and also to other geometrical constructions, such as moment maps for symplectic structures or conservation laws. We apply these results to the equivariant characteristic forms on the bundle of connections, and to the equivariant Pontryagin forms on the bundle of Riemmanian metrics.

1. Introduction

The principal aim of this contribution is to show the necessity of using equivariant cohomology in the theory of jet bundles and variational bicomplexes, as well as to present some results obtained in that way.

Let $p\colon E \to M$ be a fibred manifold. We start by considering a map

$$\digamma\colon \Omega^{n+k}(J^\infty E) \to \Omega^k(\Gamma(E))$$

, which allows us to interpret the differential forms on the jet bundle of degree greater than the dimension of the base manifold as differential forms on the space of global sections. We show that the analysis of this map leads in a natural way to the consideration of the functional forms of the variational bicomplex theory.

As the map $\digamma$ commutes with the exterior differential, we have an induced map in cohomology $\digamma\colon H^{n+k}(E) \cong H^{n+k}(J^\infty E) \to H^k(\Gamma(E))$, which can be used to obtain cohomology classes on $\Gamma(E)$. The space of sections and its cohomology are interesting objects of study, but in general it is more interesting the quotient space $\Gamma(E)/\mathcal{G}$ of sections modulo the action of a symmetry group of transformations $\mathcal{G}$, and some subspaces of this quotient, usually moduli spaces of solutions to certain differential equations (anti-self-dual or Yang-Mills connections, Einstein metrics, J-holomorphic curves, harmonic maps, etc.). In fact, cohomology classes on moduli spaces are used in the definition, for example, of Donaldson or Gromov-Witten invariants (***e.g.*** see [**8**], [**15**]).

Taking this considerations into account, we are led to try to extend our constructions in order to obtain differential forms and cohomology classes on the quotient space $\Gamma(E)/\mathcal{G}$ by working on the jet bundle $J^\infty E$. We show this is possible by using equivariant cohomology.

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The map $\mathcal{F}$ easily extends to a map between equivariant differential forms and induces a map in equivariant cohomology $\mathcal{F}\colon H^{n+k}_{\mathcal{G}}(J^\infty E) \to H^k_{\mathcal{G}}(\Gamma(E))$. The equivariant cohomology classes on $\Gamma(E)$ obtained in this way can be used to define cohomology classes on the quotient $\Gamma(E)/\mathcal{G}$ in the case of free actions (see below for details).

Furthermore, the equivariant differential forms provide more information than the cohomology classes on the quotient $\Gamma(E)/\mathcal{G}$. For example, by the well-known relationship between equivariant symplectic forms and moment maps (see [**4**]), an equivariant closed $(n+2)$-form on $J^\infty E$ defines a pre-symplectic structure on $\Gamma(E)$, and a moment map for it. Moreover, we show that equivariant closed $(n+1)$-forms on $J^\infty E$ provide conserved currents for certain locally variational operators. Notice that the existence of conserved currents is not related to the cohomology of $\Gamma(E)/\mathcal{G}$, but both of them are provided by the equivariant cohomology of the jet bundle.

The preceding constructions furnish general abstract results about equivariant differential $(n+k)$-forms on the jet bundle of a given fibre bundle. Altough they are not very common objects of study, applications of them can be given to interesting examples. Below, we consider two important instances: connections on principal bundles and Riemannian metrics, where equivariant differential forms of this type (the equivariant characteristic and Pontryagin forms) appear in a natural way, and we comment how the application of our constructions to these forms leads to important objects on gauge theories and differential geometry.

2. Forms on $\Gamma(E)$ induced by forms on $J^\infty E$

Let $p\colon E \to M$, be a bundle over a compact, oriented and without boundary n-manifold M.

We denote by J^rE its r-jet bundle, and by $J^\infty E$ the infinite jet bundle (see [**1, 16**] for the details on the geometry of $J^\infty E$).

We denote by $\mathrm{Proj}E$ the space of projectable diffeomorphism of E, ***i.e.*** the $\varphi \in \mathrm{Diff}E$ such that there exist $\underline{\varphi} \in \mathrm{Diff}M$ satisfying $\varphi \circ p = p \circ \underline{\varphi}$. We denote by Proj^+E the subgroup of elements such that $\underline{\varphi} \in \mathrm{Diff}^+M$, ***i.e.*** $\underline{\varphi}$ is orientation preserving. The space of projectable vector fields on E is denoted by $\mathrm{proj}E$, and can be considered as the Lie algebra of $\mathrm{Proj}E$. We denote by $\varphi^{(\infty)}$ (resp. $\mathrm{pr}X$) the prolongation of $\varphi \in \mathrm{Proj}E$ (resp. $X \in \mathrm{proj}E$) to $J^\infty E$.

Let $\Gamma(E)$ be the space of global sections of E, and assume that it is non-empty. We consider $\Gamma(E)$ as a differential manifold; for the details of its infinite-dimensional structure see [**14**].

For any $s \in \Gamma(E)$, the tangent space to the manifold $\Gamma(E)$ is isomorphic to the space of vertical vector fields along s, that is

$$T_s\Gamma(E) \simeq \Gamma(M, s^*V(E)). \tag{1}$$

We consider the map

$$\begin{aligned} \mathrm{j}^\infty\colon M \times \Gamma(E) &\to J^\infty E, \\ (x,s) &\longmapsto j^\infty_x s. \end{aligned}$$

Definition 1. *We define a map* $\mathcal{F}\colon \Omega^{n+k}(J^\infty E) \longrightarrow \Omega^k(\Gamma(E))$, *by*

$$\mathcal{F}[\alpha] = \int_M (\mathrm{j}^\infty)^* \alpha, \quad \alpha \in \Omega^{n+k}(J^\infty E).$$

The explicit expression of the form $\digamma[\alpha]$, using the isomorphism (1) is the following (see [**9**])

PROPOSITION 1. *Let $\alpha \in \Omega^{n+k}(J^\infty E)$ and $X_1, \dots, X_k \in T_s\Gamma(E)$. Then we have*

$$\digamma[\alpha]_s(X_1, \dots, X_k) = \int_M (j^\infty s)^*(i_{\mathrm{pr}X_k} \dots i_{\mathrm{pr}X_1}\alpha).$$

The group Proj$E$ acts naturally on $\Gamma(E)$ in the following way: if $\varphi \in \mathrm{Proj}E$, we define $\varphi_{\Gamma(E)} \in \mathrm{Diff}\Gamma(E)$ by $\varphi_{\Gamma(E)}(s) = \varphi \circ s \circ \underline{\varphi}^{-1}$, for all $s \in \Gamma(E)$. In a similar way, a projectable vector field $X \in \mathrm{proj}E$ induces a vector field $X_{\Gamma(E)} \in \mathfrak{X}(M)$.

PROPOSITION 2. *The operator $\digamma$ satisfies the following properties*

(1) For any $\alpha \in \Omega^{n+k}(E)$ we have $\digamma[d\alpha] = \mathrm{d}\digamma[\alpha]$.
(2) $\digamma[(\varphi^{(\infty)})^*\alpha] = \varphi^*_{\Gamma(E)}\digamma[\alpha]$, for any $\varphi \in \mathrm{Proj}^+E$.
(3) $\digamma[i_{\mathrm{pr}X}\alpha] = \mathrm{i}_{X_{\Gamma(E)}}\digamma[\alpha]$ for any $X \in \mathrm{proj}E$

3. The map $\digamma$ and the Variational Bicomplex

First we recall the basic definitions of the variational bicomplex theory (see [**1**] for details). On $J^\infty E$ we have a bigraduation $\Omega^k(J^\infty E) = \bigoplus_{k=p+q}\Omega^{p,q}(J^\infty E)$ into horizontal and contact degree. The exterior differential decomposes according to this bigraduation $d = d_H + d_V$. If $\alpha \in \Omega^k(J^\infty E)$ we denote by $\alpha_{p,q} \in \Omega^{p,q}(J^\infty E)$ its p-horizontal and q-contact component. We denote $\Omega^{p,q}(J^\infty E)$ simply by $\Omega^{p,q}$ when there is no risk of confusion.

We denote by $I\colon \Omega^{n,k} \to \Omega^{n,k}$, $k > 0$, the integration by parts operator. We recall that it satisfies the following properties: $I^2 = I$, $\ker I = d_H(\Omega^{n-1,k})$. The image of the integration by parts operator $\mathcal{F}^k = I(\Omega^{n,k})$ is called the space of functional k-forms. We have $\Omega^{n,k} \cong \mathcal{F}^k \oplus d_H(\Omega^{n-1,k})$.

The exterior differential induces a differential in the space of functional forms $\delta\colon \mathcal{F}^k \to \mathcal{F}^{k+1}$, $\delta\alpha = I(d_V\alpha)$.

We have the usual diagram

$$\begin{array}{cccccccccccc}
\uparrow d_V & & & & & & \uparrow d_V & & \uparrow d_V & & \uparrow \delta \\
\Omega^{0,3} & & & & & & \Omega^{n-1,3} & \xrightarrow{d_H} & \Omega^{n,3} & \xrightarrow{I} & \mathcal{F}^3 \\
\uparrow d_V & & \uparrow d_V & & & & \uparrow d_V & & \uparrow d_V & & \uparrow \delta \\
\Omega^{0,2} & \xrightarrow{d_H} & \Omega^{1,2} & \xrightarrow{d_H} & \dots & \xrightarrow{d_H} & \Omega^{n-1,2} & \xrightarrow{d_H} & \Omega^{n,2} & \xrightarrow{I} & \mathcal{F}^2 \\
\uparrow d_V & & \uparrow d_V & & & & \uparrow d_V & & \uparrow d_V & & \uparrow \delta \\
\Omega^{0,1} & \xrightarrow{d_H} & \Omega^{1,1} & \xrightarrow{d_H} & \dots & \xrightarrow{d_H} & \Omega^{n-1,1} & \xrightarrow{d_H} & \Omega^{n,1} & \xrightarrow{I} & \mathcal{F}^1 \\
\uparrow d_V & & \uparrow d_V & & & & \uparrow d_V & & \uparrow d_V & \nearrow \delta & \\
\Omega^{0,0} & \xrightarrow{d_H} & \Omega^{1,0} & \xrightarrow{d_H} & \dots & \xrightarrow{d_H} & \Omega^{n-1,0} & \xrightarrow{d_H} & \Omega^{n,0} & &
\end{array}$$

The Euler-Lagrange complex $\mathcal{E}^\bullet(J^\infty E)$ is the following complex

$$\Omega^{0,0} \xrightarrow{d_H} \Omega^{1,0} \xrightarrow{d_H} \dots \xrightarrow{d_H} \Omega^{n-1,0} \xrightarrow{d_H} \Omega^{n,0} \xrightarrow{\delta} \mathcal{F}^1 \xrightarrow{\delta} \mathcal{F}^2 \xrightarrow{\delta} \cdots$$

It is a classical result (see ***e.g.*** [**1, 17**]) that

$$H^\bullet(\mathcal{E}^\bullet(J^\infty E)) \cong H^\bullet(J^\infty E) \cong H^\bullet(E). \tag{2}$$

Now we study the relation between the map $\digamma$ and the variational bicomplex.

THEOREM 1. *For $\alpha \in \Omega^{n+k}(J^\infty E)$, $k>0$, we have*

$$\digamma[\alpha] = \digamma[\alpha_{n,k}] = \digamma[I(\alpha_{n,k})].$$

Moreover

$$\digamma[\alpha] = 0 \Leftrightarrow I\left(\alpha_{n,k}\right) = 0.$$

PROOF. That $\digamma[\alpha] = \digamma[\alpha_{n,k}]$ easily follows form Proposition 1. As we have $\alpha_{n,k} = I(\alpha_{n,k}) + d_H\eta$ for some $\eta \in \Omega^{n-1,k}$, to prove that $\digamma[\alpha_{n,k}] = \digamma[I(\alpha_{n,k})]$ it is enough to prove that $\digamma[d_H\eta] = 0$. We have

$$\begin{aligned}
\digamma\left[d_H\eta\right]_s(X_1,\dots,X_r) &= \int_M (j^\infty s)^*(i_{\mathrm{pr}X_k}\dots i_{\mathrm{pr}X_1}d_H\alpha)\\
&= (-1)^k \int_M (j^\infty s)^* d_H(i_{\mathrm{pr}X_k}\dots i_{\mathrm{pr}X_1}\alpha)\\
&= (-1)^k \int_M d\left((j^\infty s)^*(i_{\mathrm{pr}X_k}\dots i_{\mathrm{pr}X_1}\alpha)\right)\\
&= 0.
\end{aligned}$$

where we have used that for any vertical vector field X, the following equation holds:

$$i_{\mathrm{pr}X}d_H + d_H i_{\mathrm{pr}X} = 0.$$

The final result of the proposition follows from [**1**, Prop. 3.1]. □

Hence the map $\digamma$ is uniquely determined by its restriction to the space of functional forms $\mathcal{F}^k(J^\infty E)$ and this restriction $\mathcal{F}^k(J^\infty E) \hookrightarrow \Omega^k(\Gamma(E))$ is injective.

COROLLARY 1. *For $\alpha \in \Omega^{n+k}(J^\infty E)$ we have*

$$\begin{aligned}
\mathrm{d}\digamma[\alpha] &= \digamma\left[\delta(\alpha_{n,k})\right],\\
\mathrm{d}(\digamma[\alpha]) = 0 &\Longleftrightarrow \delta(\alpha_{n,k}) = 0.
\end{aligned}$$

EXAMPLE. *Let $\lambda \in \Omega^{n,0}(J^\infty E)$ be a Lagrangian density, and $\mathcal{E}(\lambda) = \delta\lambda \in \Omega^{n+1}(J^\infty E)$ be its Euler-Lagrange operator. Then $\digamma[\lambda] \in \Omega^0(\Gamma(E))$ is its action functional*

$$\digamma[\lambda](s) = \int_M (j^\infty s)^*\lambda,$$

and we have $\mathrm{d}\digamma[\lambda] = \digamma[d\lambda] = \digamma[\mathcal{E}(\lambda)]$.

The usefulness of the Euler-Lagrange operator is that of being a ***differential operator*** that characterizes the critical sections of the Lagrangian, the variationally trivial Lagrangians, etc. Our constructions generalize these properties to higher order differential forms. The integration by parts operator (which is a differential operator) allows us to know whether a differential form on the infinite dimensional manifold $\Gamma(E)$ is zero or not, closed or invariant by means of working on $J^\infty E$, where local coordinates can be used to make computations.

We have an induced map in cohomology

$$H^{n+k}(E) \cong H^k(\mathcal{F}^\bullet(J^\infty E), \delta) \to H^k(\Gamma(E)).$$

This map is not injective in general:

Example. *Let* $\pi_1\colon E = S^2 \times T^3 \to S^2$ *where* $T^3 = S^1 \times S^1 \times S^1$, *and consider* $\alpha = \pi_2^*(\omega)$ *where* $\omega = dx \wedge dy \wedge dz$ *is the standard volume form on* T^3.

We have $[\alpha] \neq 0$ *in* $H^3(E)$ *but* $\digamma[\alpha] \in \Omega^1(\Gamma(E))$ *is exact:*

As $H^1(S^2) = 0$ *for any* $\varphi\colon S^2 \to T^3$ *we have* $\varphi^*(dx) = da_\varphi$ *for some* $a_\varphi \in \Omega^0(S^2)$.

We define $G \in \Omega^0(\Gamma(E))$ *by*

$$G(\varphi) = \int_{S^2} a_\varphi \varphi^*(dy \wedge dz).$$

It is easy to see that G *is independent of the* a_φ *chosen and that we have* $\digamma[\alpha] = \mathrm{d}G$.

In order to understand the meaning of the cohomology of the space of functional forms, we introduce the following

Definition 2. *The space* $\Omega^k_{\mathrm{loc}}(\Gamma(E))$ *of local* k*-forms on* $\Gamma(E)$, *is the image of the map* $\digamma$

$$\Omega^k_{\mathrm{loc}}(\Gamma(E)) = \digamma(\Omega^{n+k}(J^\infty E)) \subset \Omega^k(\Gamma(E)).$$

By virtue of Corollary 1, $\Omega^\bullet_{\mathrm{loc}}(\Gamma(E))$ *is closed under* d, *and we denote by* $H^\bullet_{\mathrm{loc}}(\Gamma(E))$ *the cohomology of this complex.*

By Theorem 1, $\digamma$ induces a bijective correspondence between the space of local forms on $\Gamma(E)$ and the space of functional forms on the jet bundle. Furthermore, $\digamma$ induces an isomorphism $H^k(\mathcal{F}^\bullet(J^\infty E), \delta) \cong H^k_{\mathrm{loc}}(\Gamma(E))$.

Assume now that a Lie group $\mathcal{G}$ acts on E by projectable diffeomorphisms. Then we can consider the $\mathcal{G}$-invariant variational bicomplex (see [**1, 2**]). We denote by $\Omega^\bullet(J^\infty E)^{\mathcal{G}}$ (resp. $\mathcal{F}^\bullet(J^\infty E)^{\mathcal{G}}$) the space of $\mathcal{G}$-invariant forms (resp. functional forms) on $J^\infty E$, and by $\Omega^k_{\mathrm{loc}}(\Gamma(E))^{\mathcal{G}}$ the space of local $\mathcal{G}$-invariant forms, ***i.e.*** the invariant elements under the action of $\mathcal{G}$ on $\Omega^k_{\mathrm{loc}}(\Gamma(E))$. By Theorem 1 and Proposition 2, $\digamma$ induces an isomorphism $\Omega^k_{\mathrm{loc}}(\Gamma(E))^{\mathcal{G}} \cong \mathcal{F}^k(J^\infty E)^{\mathcal{G}}$, and hence the local $\mathcal{G}$-invariant cohomology is determined by the cohomology of the $\mathcal{G}$-invariant Euler-Lagrange bicomplex $H^k(\mathcal{F}^\bullet(J^\infty E)^{\mathcal{G}}, \delta) \cong H^k_{\mathrm{loc}}(\Gamma(E))^{\mathcal{G}}$. Under very general conditions we have (see [**1**]) $H^r(\mathcal{E}^\bullet(J^\infty E)^{\mathcal{G}}) \cong H^r(J^\infty E)^{\mathcal{G}}$, and in that case we have $H^k_{\mathrm{loc}}(\Gamma(E))^{\mathcal{G}} \cong H^{n+k}(J^\infty E)^{\mathcal{G}}$.

4. Equivariant Cohomology

In this section we recall the definition of equivariant cohomology and its relation with the cohomology of the quotient space. Recall first that if a Lie group $\mathcal{G}$ acts on a manifold N, we can consider several different cohomologies: the ordinary cohomology $H^\bullet(N)$ of N, the $\mathcal{G}$-invariant cohomology $H^\bullet(N)^{\mathcal{G}}$ (***i.e.***, the cohomology of the complex of $\mathcal{G}$-invariant differential forms on N) and the $\mathcal{G}$-equivariant cohomology $H^\bullet_{\mathcal{G}}(N)$ of N (defined bellow).

If $\mathcal{G}$ is compact there is an isomorphism $H^\bullet(N)^{\mathcal{G}} \cong H^\bullet(N)$. If in addition the action is free, a classical theorem by H. Cartan states that we have an isomorphism (see [**13**]) $H^\bullet_{\mathcal{G}}(N) \cong H^\bullet(N/\mathcal{G})$.

In the non-compact case, the equivariant cohomology is still related to the cohomology of the quotient, as we will see.

We recall the definition of equivariant de Rham cohomology in the Cartan model (***e.g.*** see [**6, 13**]). Suppose that we have a left action of a connected Lie group $\mathcal{G}$ on a manifold N, ***i.e.*** a homomorphism $\rho\colon \mathcal{G} \to \mathrm{Diff}(N)$. We have an induced Lie algebra homomorphism $\mathrm{Lie}\,\mathcal{G} \to \mathfrak{X}(N)$, $X \mapsto X_N$.

We denote by $\mathcal{P}^k(\mathrm{Lie}\,\mathcal{G}, \Omega^r(N))^{\mathcal{G}}$ the space of degree k $\mathcal{G}$-invariant polynomials on $\mathrm{Lie}\,\mathcal{G}$ with values in $\Omega^r(N)$. We assign to its elements degree $2k+r$. The space of $\mathcal{G}$-equivariant differential q-forms is

$$\Omega^q_{\mathcal{G}}(N) = \bigoplus_{2k+r=q} (\mathcal{P}^k(\mathrm{Lie}\,\mathcal{G}, \Omega^r(N)))^{\mathcal{G}}.$$

The Cartan differential $d_c\colon \Omega^q_{\mathcal{G}}(N) \to \Omega^{q+1}_{\mathcal{G}}(N)$ is defined by

$$(d_c\alpha)(X) = d(\alpha(X)) - i_{X_N}\alpha(X), \quad \forall X \in \mathrm{Lie}\,\mathcal{G}.$$

As is well known, on $\Omega^\bullet_{\mathcal{G}}(N)$ we have $(d_c)^2 = 0$. Moreover, the equivariant cohomology $H^q_{\mathcal{G}}(N)$ (in the Cartan model) of N with respect to the action of $\mathcal{G}$ is defined as the cohomology of this complex.

Next, let us recall the relationship between equivariant cohomology and the cohomology of the quotient space. If the action of $\mathcal{G}$ on N is free and $N/\mathcal{G}$ is a manifold, then $N \to N/\mathcal{G}$ is a principal $\mathcal{G}$-bundle. Then there is a map (called the generalized Chern-Weil homomorphism)

$$\mathrm{ChW}_N\colon H^\bullet_{\mathcal{G}}(N) \to H^\bullet(N/\mathcal{G}).$$

If $\alpha = f_i \otimes \beta^i \in \Omega^\bullet_{\mathcal{G}}(N)$, with $f_i \in \mathcal{P}^\bullet(\mathrm{Lie}\,\mathcal{G})$ and $\beta^i \in \Omega^\bullet(N)$, and A is a connection on the principal $\mathcal{G}$-bundle $N \to N/\mathcal{G}$, we have

$$\mathrm{ChW}_N([\alpha]) = [f_i(F_A) \wedge \beta^i_{\mathrm{hor}}].$$

EXAMPLES

(1) If $\alpha \in \Omega^q(N)^{\mathcal{G}}$, then $\alpha \in \Omega^q_{\mathcal{G}}(N)$ and

$$d_c\alpha = 0 \;\Leftrightarrow\; \begin{cases} d\alpha = 0, \\ i_{X_N}\alpha = 0, \; \forall X \in \mathfrak{g}. \end{cases}$$

(2) If $f \in \mathcal{I}^{\mathcal{G}}_k = \mathcal{P}^k(\mathfrak{g}, \mathbb{R})^{\mathcal{G}}$ then $f \in \Omega^{2k}_{\mathcal{G}}(N)$ and $d_c f = 0$.

(3) If $\alpha \in \Omega^2_{\mathcal{G}}(N)$ then $\alpha = \omega + \mu$ where $\omega \in \left(\mathcal{P}^0(\mathfrak{g}, \Omega^2(N))\right)^{\mathcal{G}} \cong \Omega^2(N)^{\mathcal{G}}$, and $\mu \in \left(\mathcal{P}^1(\mathfrak{g}, \Omega^0(N))\right)^{\mathcal{G}} \cong \mathrm{Hom}\,(\mathfrak{g}, C^\infty(N))^{\mathcal{G}}$, ***i.e.*** μ is a $\mathcal{G}$-equivariant linear map $\mu\colon \mathfrak{g} \to C^\infty(N)$. We have

$$d_c\alpha = 0 \;\Leftrightarrow\; \begin{cases} d\omega = 0, \\ i_{X_N}\omega = d(\mu(X)), \; \forall X \in \mathfrak{g}. \end{cases}$$

A closed 2-form ω is called a pre-symplectic form. If it is non-degenerated, it is called a symplectic form. If $\mathcal{G}$ acts on N and ω is $\mathcal{G}$-invariant, a moment map for ω is an equivariant linear map $\mu\colon \mathfrak{g} \to C^\infty(N)$ such that for all $X \in \mathfrak{g}$ $i_{X_N}\omega = d(\mu(X))$. A $\mathcal{G}$-equivariant closed 2-form is called a $\mathcal{G}$-equivariant pre-symplectic form. Hence we obtain that ***a $\mathcal{G}$-equivariant pre-symplectic form is the same as a $\mathcal{G}$-invariant pre-symplectic form and a moment map for it.***

5. Extension of $\digamma$ to equivariant cohomology

The map $\digamma$ extends to a map $\digamma\colon \Omega^{n+k}_{\mathcal{G}}(J^\infty E) \to \Omega^k_{\mathcal{G}}(\Gamma(E))$ between equivariant differential forms in the following way (see [**9**]). If $\alpha \in \Omega^{n+k}_{\mathcal{G}}(J^\infty E)$, then we define $\digamma[\alpha](X) = \digamma[\alpha(X)]$, for any $X \in \mathrm{Lie}\;\mathcal{G}$, that is, $\digamma[\alpha]$ is the composition

$$\mathfrak{g} \xrightarrow{\alpha} \Omega^{n+r}(J^\infty E) \xrightarrow{\digamma} \Omega^r(\Gamma(E)).$$

Proposition 3. *For any* $\alpha \in \Omega^{n+k}_{\mathcal{G}}(J^\infty E)$ *we have* $\digamma[d_c\alpha] = \mathrm{d}_c\digamma[\alpha]$.

We define the space of $\mathcal{G}$-equivariant functional forms

$$\mathcal{F}^k_{\mathcal{G}}(J^\infty E) = \bigoplus\nolimits_{2k+r=q}(\mathcal{P}^k(\mathfrak{g}, \mathcal{F}^r(J^\infty E)))^{\mathcal{G}}$$

and a differential $\delta_c \colon \mathcal{F}^k_{\mathcal{G}}(J^\infty E) \to \mathcal{F}^{k+1}_{\mathcal{G}}(J^\infty E)$ by $(\delta_c\alpha)(X) = I(d_c\alpha(X))$.

Remark 1. *It would be natural to consider an analogous definition for the horizontal differential forms in order to obtain an "equivariant Euler-Lagrange complex" analogous to the ordinary one. However, the situation is not so simple, as the "horizontal Cartan differential" that we obtain by restricting the Cartan differential to horizontal forms does not square to zero.*

We have an injective map $\digamma \colon \mathcal{F}^k_{\mathcal{G}}(J^\infty E) \hookrightarrow \Omega^k_{\mathcal{G}}(\Gamma(E))$ and a (non-injective in general) map in equivariant cohomology $\digamma \colon H^k(\mathcal{F}^\bullet_{\mathcal{G}}(J^\infty E), \delta_c) \to H^k_{\mathcal{G}}(\Gamma(E))$.

Definition 3. *We define the space* $\Omega^k_{\mathcal{G},\mathrm{loc}}(\Gamma(E))$ *of local* $\mathcal{G}$*-equivariant forms as*

$$\Omega^k_{\mathcal{G},\mathrm{loc}}(\Gamma(E)) = \digamma(\Omega^{n+k}_{\mathcal{G}}(J^\infty E)) \subset \Omega^k_{\mathcal{G}}(\Gamma(E)).$$

$\Omega^\bullet_{\mathcal{G},\mathrm{loc}}(\Gamma(E))$ *is closed under* d_c*, and its cohomology is denoted by* $H^k_{\mathcal{G},\mathrm{loc}}(\Gamma(E))$

Again we have isomorphisms $\mathcal{F}^\bullet_{\mathcal{G}}(J^\infty E) \cong \Omega^k_{\mathcal{G},\mathrm{loc}}(\Gamma(E))$, and

$$H^k(\mathcal{F}^\bullet_{\mathcal{G}}(J^\infty E), \delta_c) \cong H^k_{\mathcal{G},\mathrm{loc}}(\Gamma(E)).$$

6. Applications

6.1. Cohomology classes on $\Gamma(E)/\mathcal{G}$ **.** Assume now that $\mathcal{G}$ acts freely on $\Gamma(E)$ and that we have a well defined quotient manifold $\Gamma(E)/\mathcal{G}$. In this case, composing $\digamma$ and $\mathrm{ChW}_{\Gamma(E)}$ we obtain a map

$$H^{n+k}_{\mathcal{G}}(J^\infty E) \xrightarrow{\digamma} H^k_{\mathcal{G}}(\Gamma(E)) \xrightarrow{\mathrm{ChW}_{\Gamma(E)}} H^k(\Gamma(E)/\mathcal{G}).$$

Hence, closed equivariant forms on $J^\infty E$ determine cohomology classes on the quotient space $\Gamma(E)/\mathcal{G}$.

6.2. Pre-symplectic forms and moment maps. If $\alpha \in \Omega^{n+2}_{\mathcal{G}}(J^\infty E)$ and $d_c\alpha = 0$, then $\digamma[\alpha] \in \Omega^2_{\mathcal{G}}(\Gamma(E))$ is a $\mathcal{G}$-equivariant pre-symplectic form, and hence determines a pre-symplectic structure on $\Gamma(E)$ and a moment map for it. We show below examples where closed equivariant $(n+2)$-forms appear in a natural way, and how the corresponding pre-symplectic forms and moment maps are related to classical objects in differential geometry.

6.3. Locally variational operators and conserved currents.

If $\alpha \in \Omega^{n+1}_{\mathcal{G}}(J^\infty E)$ and $d_c\alpha = 0$, then $\digamma[\alpha] \in \Omega^1_{\mathcal{G}}(\Gamma(E))$ determines a basic 1-form on $\Gamma(E)$.

Furthermore, we have $\alpha(X) = \alpha(0) + \mu(X) + \dots$ with

$$\alpha(0) \in \Omega^{n+1}(J^\infty E)^{\mathcal{G}},$$
$$\mu \colon \mathfrak{g} \to \Omega^{n-1}(J^\infty E).$$

The form $\Omega = I(\alpha(0)_{n,1}) \in \mathcal{F}^1$ is a $\mathcal{G}$-invariant locally variational operator, and we have the conserved current for Ω

$$\begin{gathered}\mathfrak{J}\colon \mathfrak{g} \to \Omega^{n-1}(J^\infty E)\\ \mathfrak{J}(X) = \mu(X) + i_{\mathrm{pr}X}\eta,\end{gathered}$$

where $\eta \in \Omega^{n-1,1}(J^\infty E)$ satisfies $\alpha(0)_{n,1} = \Omega + d_H\eta$.

Hence, if we have a closed $\mathcal{G}$-equivariant differential $(n+1)$-form on $J^\infty E$, we obtain a locally variational operator Ω and a conserved current $\mathfrak{J}$ for it. This applies not only to the classical case were we have a $\mathcal{G}$-invariant Lagrangian, but also to the case were there are no invariant Lagrangians (or no Lagrangian at all) for Ω. One easy example comes from classical mechanics, were conserved currents are related to moment maps, ***i.e.***, to equivariant differential forms. See below for other examples.

7. Examples

7.1. Characteristic forms on the bundle of connections. Let $\pi\colon P \to M$ a principal G-bundle over a compact n-manifold M, and let $p\colon C(P) \to M$, be its bundle of connections, ***i.e.*** a bundle whose global sections are connections on P (for the definition and properties of the bundle of connections, see for example [**7**]). For any Weil polynomial $f \in \mathcal{I}^G_k$ of degree k for the group G we have (see [**9**])

- The characteristic form $c_f(\mathbb{F}) \in \Omega^{2k}(C(P))$,

- The equivariant characteristic form $c_f(\mathbb{F}^{\mathrm{Aut}P}) \in \Omega^{2k}_{\mathrm{Aut}P}(C(P))$.

We can restrict the group $\mathrm{Aut}P$ to any subgroup. We will mainly consider the subgroup of gauge transformations $\mathrm{Gau}P$.

• Let $f \in \mathcal{I}^G_k$, and let $\beta \in \Omega^r(M)$ be a closed form. If $2k + r > n$, then $\fint[c_f(\mathbb{F}^{\mathrm{Gau}P}) \wedge p^*\beta] \in \Omega^{2k+r-n}_{\mathrm{Gau}P}(\mathcal{A})$ is a closed $\mathrm{Gau}P$-equivariant form on the space of connections $\mathcal{A}$. Let $\mathrm{Gau}^0P \subset \mathrm{Gau}P$ be the subgroup of gauge transformations acting as the identity on the fibre of a fixed point $x_0 \in M$. Then Gau^0P acts freely on the space of connections $\mathcal{A}$, and by the constructions of Subsection 6.1 we obtain cohomology classes on the quotient space $\mathcal{A}/\mathrm{Gau}^0P$, which are proved (see [**9**]) to coincide with the classes defined by Atiyah and Singer in [**5**].

Now let $G = SU(2)$ and $f(X) = -\frac{1}{8\pi^2}\mathrm{tr}(X^2)$ the polynomial corresponding to the second Chern class

• If $n = 2$, then $c_f(\mathbb{F}) \in \Omega^4(C(P))$ and $\omega = \fint[c_f(\mathbb{F})] \in \Omega^2(\mathcal{A})$ is (up to an scalar factor) the natural symplectic structure on $\mathcal{A}$ defined in [**3**],

$$\omega_A(a,b) = -\tfrac{1}{4\pi^2}\int_M \mathrm{tr}(a \wedge b), \quad a, b \in \Omega^1(M, \mathrm{ad}P),$$

and the corresponding moment map is given by

$$\begin{gathered}\mu\colon \mathrm{gau}P = \Omega^0(M, \mathrm{ad}P) \to \Omega^0(\mathcal{A})\\ \mu(X)_A = \tfrac{1}{4\pi^2}\int_M \mathrm{tr}(F_A \cdot X), \quad X \in \mathrm{gau}P.\end{gathered}$$

and it coincides with the one defined in [**3**].

Hence: ***The second Chern form of an $SU(2)$-bundle over a surface defines the natural symplectic structure on the space of connections, and the second equivariant Chern form determines the natural moment map for it.***

In [3] the corresponding symplectic reduction is studied, and the result is that the Marsden-Weinstein quotient is the moduli space of flat connections, and in this way a canonical symplectic structure on this space is obtained. In the next section we will study the analogous situation for the space of Riemannian metrics.

• If $n = 3$, then $c_f(\mathbb{F}) \in \mathcal{F}^1(J^\infty C(P))$ is the Euler-Lagrange operator of the Chern-Simons Lagrangian (see [18]). According to Subsection 6.3 the second equivariant Chern form determines a conserved current, that is given by

$$\mathfrak{J}(X) = \tfrac{1}{4\pi^2}\mathrm{tr}(\mathbb{F}\cdot X), \quad X \in \mathrm{gau}P,$$

and coincides with the one defined in [18].

Note that the Chern-Simons Lagrangian is not Gau$P$-invariant, but also in this case we have a conserved current, provided by the second equivariant Chern form.

7.2. Pontryagin forms on the bundle of Riemannian metrics. Let $\mathcal{M}_M \to M$ be the bundle of Riemannian metrics of the n-manifold M and $\mathcal{G} = \mathrm{Diff}^+(M)\times\mathbb{R}$.

A construction, similar to the preceding one for connections in principal bundles but for Riemannian metrics, can be given to define on $J^1\mathcal{M}_M$ the following (see [11, 12]):

- The Pontryagin forms $p_k(\mathbf{\Omega}) \in \Omega^{4k}(J^1\mathcal{M}_M)$, $k \in \mathbb{N}$, and

- The $\mathcal{G}$-equivariant Pontryagin forms $p_k(\mathbf{\Omega}^{\mathcal{G}}) \in \Omega^{4k}_{\mathcal{G}}(J^1\mathcal{M}_M)$.

• By the considerations of Subsection 6.1, the equivariant Pontryagin classes (of degree higher than n) determine canonical cohomology classes on the space $\mathfrak{Met}_M/\mathrm{Diff}^0 M$, where $\mathfrak{Met}_M$ is the space of Riemannian metrics on M, $x_0 \in M$ and $\mathrm{Diff}^0 M = \{\varphi \in \mathrm{Diff}\, M : \varphi(x_0) = x_0, (\varphi_*)_{x_0} = \mathrm{id}\}$ acts freely on $\mathfrak{Met}_M$.

• If $n = 2$, $\omega = \digamma[p_1(\mathbf{\Omega})] \in \Omega^2(\mathfrak{Met}_M)$ is a pre-symplectic (not symplectic) form, and the first equivariant Pontryagin form determines a moment map for it $\mu\colon \mathfrak{X}(M)\times\mathbb{R} \to \Omega^0(\mathfrak{Met}_M)$. The corresponding symplectic reduction is studied in [10], and the result is the following

Theorem 2. *We have $\mu^{-1}(0) = \mathfrak{Met}_{\mathrm{const}}$ the space of metrics of constant curvature. If M has genus higher to 1, then the Marsden-Weinstein quotient of $(\mathfrak{Met}_M, \omega)$ is $(\mathcal{T}(M), \frac{1}{2\pi^2}\omega_{\mathrm{WP}})$ where $\mathcal{T}(M)$ is the Teichmüller space of M and ω_{WP} is the Weil-Petersson symplectic form (**e.g.** see [19] for the definition of the Weil-Petersson metric on $\mathcal{T}(M)$).*

• If $n = 3$, $\Omega = I(p_1(\mathbf{\Omega})) \in \mathcal{F}^1(J^\infty\mathcal{M}_M)$ is basically the Cotton-York tensor (or the Euler-Lagrange operator of the gravitational Chern-Simons Lagrangian). According to Subsection 6.3 the first equivariant Pontryagin form provides a conserved current for this operator.

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References

[1] I. Anderson, ***The Variational Bicomplex***, preprint.
[2] I. Anderson, J. Pohjanpelto, ***Infinite dimensional Lie algebra cohomology and the cohomology of invariant Euler-Lagrange complexes: A preliminary report***, Differential geometry and applications (Brno, 1995), 427–448, Masaryk Univ., Brno, 1996.
[3] M.F. Atiyah, R. Bott, ***The Yang-Mills equations over Riemann surfaces***, Phil. Trans. R. Soc. Lond. A **308** (1982) 523–615.
[4] —, ***The moment map and equivariant cohomology***, Topology **23** (1984), 1–28.
[5] M.F. Atiyah, I. Singer, ***Dirac operators coupled to vector potentials***, Proc. Natl. Acad. Sci. USA **81** (1984), 2597–2600.
[6] N. Berline, E. Getzler, M. Vergne, ***Heat Kernels and Dirac Operators***, Springer Verlag Berlin Heidelberg 1992.
[7] M. Castrillón López, J. Muñoz Masqué, ***The geometry of the bundle of connections***, Math. Z. **236** (2001), 797–811.
[8] S.K. Donaldson, P.B. Kronheimer, ***The Geometry of Four-Manifolds***, Oxford University Press, Oxford, UK, 1991.
[9] R. Ferreiro Pérez, ***Equivariant Characteristic forms in the bundle of connections***, preprint arXiv: math-ph/0307022. To appear in J. Geom Phys.
[10] —, ***The first Pontryagin form of a surface and the Weil Petersson symplectic form on Teichmüler space***, in preparation.
[11] R. Ferreiro Pérez, J. Muñoz Masqué, ***Natural connections on the bundle of Riemannian metrics***, in preparation.
[12] —, ***Equivariant characteristic forms in the bundle of metrics***, in preparation
[13] V. Guillemin, S. Sternberg, ***Supersymmetry and Equivariant de Rham Theory***, Springer-Verlag, Berlin Heidelberg, 1999.
[14] A. Kriegl, P. W. Michor, ***The Convenient Setting of Global Analysis***, Mathematical Surveys and Monographs, Volume 53, Amer. Math. Soc., 1997.
[15] D. McDuff, D. Salamon, ***J-holomorphic curves and quantum cohomology***, Univ. Lecture Ser. Volume 6, Amer. Math. Soc., RI, 1994.
[16] D.J. Saunders, ***The Geometry of Jet Bundles***, London Mathematical Society Lecture Notes Series **142**, Cambridge University Press, 1989.
[17] F. Takens, ***A global version of the inverse problem to the calculus of variations***, J. Differential Geometry **14**, (1979) 543–562.
[18] C. Tejero Prieto, ***Variational formulation of Chern-Simons theory for arbitrary Lie groups***, J. Geom Phys. **50** (2004), 138–161.
[19] A.J. Tromba, ***Teichmüller Theory in Riemannian Geometry***, Birkhäuser Basel, Boston, Berlin, 1992.

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