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# Prolongation of connections to vertical bundles

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Abstract. We study the prolongation of connections and pairs of connections to vertical bundles from a general point of view. For a general connection on a fibered manifold $Y \to M$ and a classical linear connection on $M$ we introduce the induced connection on the $F$-vertical bundle $V^F Y \to M$. We also study natural operators of this type and present some examples and applications.

## Introduction

Let $G$ be an arbitrary bundle functor defined on the category $\mathcal{FM}_{m,n}$ of fibered manifolds with $m$-dimensional bases and $n$-dimensional fibres and locally invertible fiber respecting mappings. Given a general connection $\Gamma$ on a fibered manifold $Y \to M$, one can construct an induced connection $\mathcal{G}(\Gamma, \nabla)$ on $GY \to M$ by means of an auxiliary higher order linear connection $\nabla$ on the base manifold $M$, [**5**]. Moreover, if the base order of $G$ is zero, then the induced connection $\mathcal{G}(\Gamma, \nabla)$ does not depend on the linear connection $\nabla$. For the $F$-vertical bundle $G = V^F$ determined by a natural bundle $F$ we obtain in such a way a connection $\mathcal{V}^F\Gamma$ on $V^F Y \to M$, which is called the $F$-vertical prolongation of $\Gamma$, [**6**].

We point out that the $F$-vertical functor $V^F$ plays an important role among all bundle functors on $\mathcal{FM}_{m,n}$. In particular, in [**2**] we have proved

Theorem 1. *A bundle functor $G$ on $\mathcal{FM}_{m,n}$ admits an $\mathcal{FM}_{m,n}$-natural operator transforming connections on $Y \to M$ into connections on $GY \to M$ if and only if $G$ is isomorphic to some $F$-vertical bundle functor $V^F$. For $V^F$ such natural operator is unique.*

Roughly speaking, $G = V^F$ are the only bundle functors on $\mathcal{FM}_{m,n}$, which admit prolongation of connections from $Y \to M$ to $GY \to M$. On the other hand, if $G$ is not isomorphic to $V^F$, then the construction of an induced connection on $GY \to M$ from a general connection $\Gamma$ on $Y \to M$ requires an auxiliary higher order linear connection $\nabla$ on $M$. In this paper we show, that even in the case of an $F$-vertical bundle $V^F$, the pair $(\Gamma, \nabla)$ induces a connection on $V^F Y \to M$ essentially dependent on $\nabla$. We point out that our geometric construction will be different from the connection $\mathcal{G}(\Gamma, \nabla)$, which was introduced in [**5**].

In this note we will study the prolongation of connections and pairs of connections to vertical bundles from a general point of view. In Section 2 we introduce a

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new geometric construction transforming a general connection $\Gamma$ on a fibered manifold $Y \to M$ and a classical linear connection $\nabla$ on $M$ into a connection $D(\Gamma, \nabla)$ on $V^F Y \to M$. More precisely, for any natural vector field $C$ on $F$ we define a connection $\mathcal{V}^{F,C}(\Gamma, \nabla)$ on $V^F Y \to M$ and then we prove that any natural operator $D$ in question is of the form $\mathcal{V}^{F,C}$ for some natural vector field $C$ on $F$. In Section 3 we present some examples and applications. As a direct consequence we generalize the result of I. Kolář and the second author from **[6]**, who have determined all finite order natural operators transforming connections on $Y \to M$ into connections on $V^F Y \to M$. In particular, we show that the finite order assumption can be omitted. The whole Section 4 will be devoted to the proof of our classification theorem.

Denote by $\mathcal{M}f_m$ the category of $m$-dimensional manifolds and their local diffeomorphisms and by $\mathcal{FM}$ the category of fibered manifolds and fiber respecting mappings. In what follows $Y \to M$ stands for $\mathcal{FM}_{m,n}$-objects and $N$ stands for $\mathcal{M}f_n$-objects. All manifolds and maps are assumed to be of the class $C^\infty$. Unless otherwise specified, we will use the terminology and notation from the book **[5]**.

## 1. Prolongation of connections to vertical bundles

We recall that the definition of the order of a bundle functor $G$ on the category $\mathcal{FM}_{m,n}$ is based on the concept of $(r,s,q)$-jet, $s \geqslant r \leqslant q$. Given two fibered manifolds $p : Y \to M$ and $\overline{p} : \overline{Y} \to \overline{M}$, we say that two $\mathcal{FM}_{m,n}$-morphisms $f, g : Y \to \overline{Y}$ with the base maps $\underline{f}, \underline{g} : M \to \overline{M}$ determine the same $(r,s,q)$-jet $j^{r,s,q}_y f = j^{r,s,q}_y g$ at $y \in Y$, $p(y) = x$, if

$$j^r_y f = j^r_y g, \quad j^s_y(f|Y_x) = j^s_y(g|Y_x), \quad j^q_x \underline{f} = j^g_x \underline{g}.$$

The space of all such $(r,s,q)$-jets will be denoted by $J^{r,s,q}(Y, \overline{Y})$, **[5]**. Further, a bundle functor $G$ on $\mathcal{FM}_{m,n}$ is said to be of order $(r,s,q)$, if $j^{r,s,q}_y f = j^{r,s,q}_y g$ implies $Gf|G_y Y = Gg|G_y Y$. In this case the integer $q$ is called the base order of $G$.

Consider a bundle functor $F$ on $\mathcal{M}f_n$ of the order $s$. The $F$-vertical functor $V^F$ is a bundle functor on $\mathcal{FM}_{m,n}$ defined by

$$V^F Y = \bigcup_{x \in M} F(Y_x), \quad V^F f = \bigcup_{x \in M} F(f_x)$$

where $f_x$ is the restriction and corestriction of $f : Y \to \overline{Y}$ over $\underline{f} : M \to \overline{M}$ to the fibers $Y_x$ and $\overline{Y}_{\underline{f}(x)}$, **[6]**. For the tangent bundle $F = T$ we will use the standard notation $V$ instead of $V^T$. Clearly, the order of $V^F$ is $(0, s, 0)$.

A projectable vector field on a fibered manifold $Y \to M$ is a fibered manifold morphism $Z : Y \to TY$ over the underlying vector field $M \to TM$ and its flow $\exp tZ$ is formed by local $\mathcal{FM}_{m,n}$-morphisms. Given a bundle functor $G$ on $\mathcal{FM}_{m,n}$, the flow prolongation of $Z$ with respect to $G$ is a vector field $\mathcal{G}Z : GY \to TGY$ defined by $\mathcal{G}Z = \frac{\partial}{\partial t}|_0 G(\exp tZ)$. By **[6]**, if $G$ has order $(r,s,q)$, then the value of $\mathcal{G}Z$ at each point of $G_y Y$ depends on $j^{r,s,q}_y Z$ only. Thus the flow prolongation of projectable vector fields can be interpreted as a map

$$\mathcal{G}_Y : GY \times_Y J^{r,s,q} TY \to TGY, \tag{1}$$

where $J^{r,s,q} TY$ denotes the space of all $(r,s,q)$-jets of projectable vector fields on $Y$. By **[6]**, $\mathcal{G}_Y$ is linear in the second factor.

Consider a general connection $\Gamma : Y \to J^1 Y$ on a fibered manifold $Y \to M$. We first recall the geometric construction of an induced connection $\mathcal{G}(\Gamma, \nabla)$ on $GY \to M$

by means of a linear $q$-th order connection $\nabla : TM \to J^q TM$ on $M$. Given a vector field $X$ on $M$, its $\Gamma$-lift is a projectable vector field $\Gamma X : Y \to TY$. By (1), the flow prolongation $\mathcal{G}(\Gamma X) : GY \to TGY$ can be interpreted as a map

$$\mathcal{G}\Gamma : GY \times_M J^q TM \to TGY, \tag{2}$$

which is linear in the second factor. By linearity, the composition

$$\mathcal{G}(\Gamma, \nabla) := \mathcal{G}\Gamma \circ (\mathrm{id}_{GY} \times_{\mathrm{id}_M} \nabla) : GY \times_M TM \to TGY \tag{3}$$

is the lifting map of a connection (denoted by the same symbol) on $GY \to M$. Clearly, if the base order of $G$ is $q = 0$, then (2) is a connection on $GY \to M$. For an $F$-vertical functor $G = V^F$ we obtain in such a way a connection on $V^F Y \to M$, which is called the $F$-vertical prolongation of $\Gamma$ and is denoted by $\mathcal{V}^F \Gamma$, [**6**]. Moreover, I. Kolář and the second author have proved, [**6**]

Proposition 1. *$\mathcal{V}^F$ is the only natural operator of finite order transforming connections on $Y \to M$ into conections on $V^F Y \to M$.*

In [**6**] there are also described some conditions under which every natural operator transforming connections to the $F$-vertical bundle $V^F$ has finite order. For example, the finite order assumption can be omitted for the vertical Weil functor $V^A$, which corresponds to the Weil functor $F = T^A$. In Section 2 we show that the finite order assumption can be omitted for all bundle functors $F$.

We point out that the construction of the vertical prolongation $\mathcal{V}^F \Gamma$ does not use any linear connection $\nabla$ on the base manifold $M$. But it is interesting to pose a question whether one can construct an induced connection on the $F$-vertical bundle $V^F Y \to M$ by means of some linear connection $\nabla$ on $M$. We present such a construction in the next section, where we study the prolongation of pairs of conections into a connection on $V^F Y \to M$ from the general point of view.

## 2. Prolongation of pairs of connections into a connection on vertical bundles

The prolongation of pairs of connections $\Gamma_1, \Gamma_2$ into a connection on vertical bundles is a geometric construction of the form

$$(\Gamma_1, \Gamma_2) \mapsto D(\Gamma_1, \Gamma_2) \tag{4}$$

where $D(\Gamma_1, \Gamma_2)$ is a general connection on $V^F Y \to M$. In [**1**] we have introduced the $(F, L)$-vertical prolongation $\mathcal{V}^{F,L}(\Gamma_1, \Gamma_2)$ of $\Gamma_1$ and $\Gamma_2$ by means of a natural linear operator $L$ transforming vector fields on $N$ into vector fields on $FN$. This is a geometric construction of the type (4), where both $\Gamma_1$ and $\Gamma_2$ are general connections on $Y \to M$. In this section we will introduce another geometric construction of the type (4), but in this case $\Gamma_2 = \nabla$ is a classical linear connection on $M$.

Let $F : \mathcal{M}f_n \to \mathcal{FM}$ be a natural bundle of order $s$ and $V^F : \mathcal{FM}_{m,n} \to \mathcal{FM}$ be the corresponding vertical modification. Suppose we have an $\mathcal{M}f_n$- natural vector field $C$ on $F$. We recall that this is a system of vector fields $C_N : FN \to TFN$ for every $n$-manifold $N$ satisfying $TFf \circ C_{N_1} = C_{N_2} \circ Ff$ for all local diffeomorphisms $f : N_1 \to N_2$. Further, let $\Gamma : Y \to J^1 Y$ be a general connection on an $\mathcal{FM}_{m,n}$-object $Y \to M$ and $\nabla : TM \to J^1 TM$ be a classical linear connection on the base manifold $M$. We are going to construct a general connection $\mathcal{V}^{F,C}(\Gamma, \nabla)$ on $V^F Y \to M$ depending canonically on $\Gamma$ and $\nabla$. Firstly, we define a fiber linear map

$$(\Gamma, \nabla)^{F,C} : V^F Y \times_M TM \to VV^F Y$$

covering the identity of $V^F Y$ as follows. Let $(u,v) \in (V^F Y \times_M TM)_x$, $x \in M$. We have $C_{Y_x} : F(Y_x) = (V^F Y)_x \to T((V^F Y)_x) = (VV^F Y)_x$. Further, let $T^\nabla : TM \times TM \to TM$ be the torsion tensor of $\nabla$ and $T_v^\nabla := T_x^\nabla(v, \cdot) \in \mathrm{End}(T_x M)$. We put

$$(\Gamma, \nabla)^{F,C}(u,v) = trace(T_v^\nabla) \cdot C_{Y_x}(u). \tag{5}$$

Obviously,

$$\mathcal{V}^{F,C}(\Gamma, \nabla) := \mathcal{V}^F \Gamma + (\Gamma, \nabla)^{F,C} : V^F Y \times_M TM \to TV^F Y \tag{6}$$

is a general connection on $V^F Y \to M$ canonically dependent on $\Gamma$ and $\nabla$.

Definition 1. The connection $\mathcal{V}^{F,C}(\Gamma, \nabla)$ is called the $(F, C)$-vertical prolongation of $(\Gamma, \nabla)$.

Clearly, the difference term $(\Gamma, \nabla)^{F,C}$ does not depend on the connection $\Gamma$. Further, from the geometric construction of $(\Gamma, \nabla)$ it follows directly

Lemma 1. *If the linear connection $\nabla$ is without torsion, then we have $\mathcal{V}^{F,C}(\Gamma, \nabla) = \mathcal{V}^F \Gamma$ for every natural vector field $C$ on $F$.*

Lemma 2.

$$(\Gamma, \nabla)^{F, k_1 C_1 + k_2 C_2} = k_1 (\Gamma, \nabla)^{F, C_1} + k_2 (\Gamma, \nabla)^{F, C_2}, \quad k_1, k_2 \in \mathbb{R}.$$

The main result of the present paper is the following classification theorem, the proof of which will be given in Section 4.

Theorem 2. *Let $F : \mathcal{M}f_n \to \mathcal{FM}$ be a natural bundle. All $\mathcal{FM}_{m,n}$-natural operators transforming a general connection $\Gamma$ on $Y \to M$ and a classical linear connection $\nabla$ on $M$ into a general conection on $V^F Y \to M$ are of the form*

$$(\Gamma, \nabla) \mapsto \mathcal{V}^{F,C}(\Gamma, \nabla)$$

*for all $\mathcal{M}f_n$-natural vector fields $C$ on $F$.*

As a direct consequence we obtain that the finite order assumption in Proposition 1 can be omitted.

Proposition 2. *$\mathcal{V}^F$ is the only natural operator transforming connections on $Y \to M$ into conections on $V^F Y \to M$.*

Proof. It suffices to choose a linear connection $\nabla$ without torsion and then apply Theorem 2. By Lemma 1 we have $\mathcal{V}^{F,C}(\Gamma, \nabla) = \mathcal{V}^F \Gamma$. □

We remark that we have presented another proof of Proposition 1' also in [**1**]. By Theorem 2, natural operators transforming pairs $(\Gamma, \nabla)$ into a connection on $V^F Y \to M$ depend on natural vector fields on $F$. By the following assertion it suffices to find the basis of such vector fields.

Proposition 3. *Let $C_1, \dots, C_r$ be the basis of natural vector fields on $F$. Then all natural operators transforming a connection $\Gamma$ on $Y \to M$ and a classical linear connection $\nabla$ on $M$ into a connection on $V^F Y \to M$ are of the form*

$$(\Gamma, \nabla) \mapsto \mathcal{V}^F \Gamma + k_1 (\Gamma, \nabla)^{F, C_1} + \dots + k_r (\Gamma, \nabla)^{F, C_r}, \quad k_i \in \mathbb{R}. \tag{7}$$

Proof. An arbitrary natural vector field $C$ on $F$ is of the form $C = k_1 C_1 + \dots + k_r C_r$, $k_i \in \mathbb{R}$. Then the assertion follows from Theorem 2 and from Lemma 2. □

## 3. Examples and applications

By [**5**], natural vector fields on a natural bundle $F$ can be interpreted as absolute natural operators transforming vector fields on $N$ into vector fields on $FN$. Further, every natural vector field on $F$ is vertical. The simplest example of a natural vector field is the classical Liouville vector field of the tangent bundle $TN$, which is tangent to the one-parameter family of all homotheties of $TN$. More generally, let $\Phi(t)$ be a smooth one-parameter family of natural transformations $F \to F$, where smoothness means that the map $\Phi(t)_N : \mathbb{R} \times FN \to FN$ is smooth for every manifold $N$. Then the formula $C_N = \frac{\partial}{\partial t}|_0 \Phi(t)_N$ defines a natural vector field $C_N : FN \to TFN$.

By the general theory, every product preserving bundle functor is a Weil functor $T^A$ determined by some Weil algebra $A$. By [**5**], all natural vector fields on $T^A N$ are the generalized Liouville vector fields $\lambda_D$ determined by all elements $D$ of the Lie algebra $\underline{\mathrm{Aut}}(A)$ of the group of all automorphisms of $A$. Moreover, the Lie algebra $\underline{\mathrm{Aut}}(A)$ coincides with the set $Der(A)$ of all derivations of $A$. We recall that a derivation of $A$ is a linear map $D : A \to A$ satisfying $D(ab) = D(a)b + aD(b)$ for all $a, b \in A$. By 42.11 of [**5**] and Theorem 2 we have

Proposition 4. *All natural operators transforming a connection $\Gamma$ on $Y \to M$ and a classical linear connection $\nabla$ on $M$ into a connection on the vertical Weil bundle $V^A Y \to M$ are of the form*

$$(\Gamma, \nabla) \mapsto \mathcal{V}^{T^A, \lambda_D}(\Gamma, \nabla)$$

*for all $D \in Der(A)$.*

Denote by $L_T$ the classical Liouville vector field of the tangent bundle $TN$. From Proposition 3 we obtain directly

Corollary 1. *All natural operators transforming a connection $\Gamma$ on $Y \to M$ and a classical linear connection $\nabla$ on $M$ into a connection on the vertical tangent bundle $VY \to M$ are of the form*

$$(\Gamma, \nabla) \mapsto \mathcal{V}\Gamma + k(\Gamma, \nabla)^{T, L_T}, \quad k \in \mathbb{R}.$$

Denote by $K^r_{k,n}$ the bundle functor over $n$-manifolds of $k$-dimensional contact elements of order $r$, $k \leqslant n$, which was introduced by C. Ehresmann, [**3**]. By 44.4 of [**5**], all natural operators transforming vector fields on $N$ into vector fields on $K^r_{k,n} N$ are constant multiples of the flow operator $\mathcal{K}^r_{k,n}$. This means that there are no natural vector fields on the bundle $K^r_{k,n}$. Thus we have

Proposition 5. *All natural operators transforming a connection $\Gamma$ on $Y \to M$ and a classical linear connection $\nabla$ on $M$ into a connection on $V^{K^r_{k,n}} Y \to M$ are of the form*

$$(\Gamma, \nabla) \mapsto \mathcal{V}^{K^r_{k,n}} \Gamma.$$

We point out that $K^r_{k,n}$ is an interesting example of a bundle functor $F$, where natural operators transforming pairs $(\Gamma, \nabla)$ into a connection on $V^F Y \to M$ do not depend on the linear conection $\nabla$. On the other hand, if $F$ is a natural vector bundle, then the homotheties $F \to F$ generate the Liouville vector field $L_F$ on $F$. This natural vector field $L_F$ determines the difference term $(\Gamma, \nabla)^{F, L_F}$, which is always included in the list (7) of natural operators from Proposition 2.

Write $T^{r*} N = J^r(N, \mathbb{R})_0$ for the space of all $r$-jets from an $n$-manifold $N$ into $\mathbb{R}$ with target 0. Since $\mathbb{R}$ is a vector space, $T^{r*} N$ is a vector bundle over $N$, which is called the $r$-th order cotangent bundle. The dual vector bundle $T^{(r)}N = (T^{r*}N)^*$ is called the $r$-th order tangent bundle, see [**5**] for more details. Clearly, for $r = 1$ we obtain the classical cotangent bundle $T^*N$ and the classical tangent bundle $TN$. The second author has proved, that the only natural vector fields on $T^{(r)}N$ are the constant multiples of the Liouville vector field $L_{T^{(r)}}$, see [**8**]. By Proposition 2 we have

Proposition 6. *All natural operators transforming a connection $\Gamma$ on $Y \to M$ and a classical linear connection $\nabla$ on $M$ into a connection on $V^{T^{(r)}}Y \to M$ are of the form*

$$(\Gamma, \nabla) \mapsto \mathcal{V}^{T^{(r)}}\Gamma + k(\Gamma, \nabla)^{T^{(r)}, L_{T^{(r)}}}, \quad k \in \mathbb{R}.$$

Define $r$ vector fields $L_1, \dots, L_r$ on $T^{r*}N$ by

$$L_s(j^r_x\gamma) = \frac{\partial}{\partial t}|_0(j^r_x\gamma + tj^r_x(\gamma)^s), \quad s = 1, \dots, r,$$

where $(\gamma)^s$ means the $s$-th power of $\gamma : N \to \mathbb{R}$, $\gamma(x) = 0$. Clearly, $L_1 = L_{T^{r*}}$ is the classical Liouville vector field of $T^{r*}N$. J. Kurek has introduced the $s$-th power natural transformation $A^r_s : T^{r*}N \to T^{r*}N$ by $A^r_s(j^r_x\gamma) = j^r_x(\gamma)^s$, [**7**]. Then the vector field $L_s$ can be also defined as a map $T^{r*}N \to T^{r*}N \oplus T^{r*}N \cong VT^{r*}N$, $w \mapsto (w, A^r_s(w))$. The second author has proved, that all natural vector fields on $T^{r*}N$ form a vector space, whose basis are exactly the vector fields $L_1, \dots, L_r$, [**9**]. Thus we have

Proposition 7. *All natural operators transforming a connection $\Gamma$ on $Y \to M$ and a classical linear connection $\nabla$ on $M$ into a connection on $V^{T^{r*}}Y \to M$ are of the form*

$$(\Gamma, \nabla) \mapsto \mathcal{V}^{T^{r*}}\Gamma + k_1(\Gamma, \nabla)^{T^{r*}, L_1} + \dots + k_r(\Gamma, \nabla)^{T^{r*}, L_r}, \quad k_i \in \mathbb{R}.$$

We remark that for many particular natural bundles $F$ there are classified natural operators transforming vector fields on $N$ into vector fields on $FN$. Then the list of such natural operators enables us to classify all natural vector fields on $F$. For example, P. Kobak has proved that all natural vector fields on the iterated bundle $TT^*$ are generated by two vector fields, [**4**]. Further, J. Tomáš has classifies all natural vector fields on the cotangent bundle of the bundle of $k$-dimensional velocities of order $r$ and also on the cotangent bundle of a Weil bundle $T^*T^A$, see [**10**] and [**11**].

## 4. Proof of Theorem 2

From now on $\mathbb{R}^{m,n}$ means the trivial bundle $\mathbb{R}^m \times \mathbb{R}^n$ over $\mathbb{R}^m$ and the usual cordinates on $\mathbb{R}^{m,n}$ will be denoted by $x^1, ..., x^m, y^1, ..., y^n$. If $\widetilde{D}$ is a natural operator of our type, then for a general connection $\Gamma$ on an $\mathcal{FM}_{m,n}$-object $Y \to M$ and a classical linear connection $\nabla$ on $M$, the difference

$$\Delta(\Gamma, \nabla) = \widetilde{D}(\Gamma, \nabla) - \mathcal{V}^F\Gamma : V^FY \times_M TM \to V(V^FY)$$

is a fiber linear map covering the identity on $V^FY$. So it remains to describe all natural operators of the type as $\Delta$. In local coordinates $(x^i, y^j)$ on $\mathbb{R}^{m,n}$, a connection $\Gamma$ has the coordinate expression

$$\Gamma = \sum_{i=1}^m dx^i \otimes \frac{\partial}{\partial x^i} + \sum_{i=1}^m \sum_{j=1}^n \Gamma^j_i dx^i \otimes \frac{\partial}{\partial y^j}$$

and a linear connection $\nabla$ can be expressed in the form

$$\nabla = \sum_{i=1}^{m} dx^i \otimes \frac{\partial}{\partial x^i} + \sum_{i,k,l=1}^{m} \nabla_{ik}^{l} dx^i X^k \otimes \frac{\partial}{\partial X^l}$$

where $X^i = dx^i$ are the induced coordinates on $T\mathbb{R}^m$. Consider a natural operator $D$ of the type as $\Delta$. We first prove some auxiliary lemmas.

LEMMA 3. *Suppose that*

$$D(\sum_{i=1}^{m} dx^i \otimes \frac{\partial}{\partial x^i} + \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{|\alpha|+|\beta|\leqslant K} \Gamma_{i\alpha\beta}^{j} x^\alpha y^\beta dx^i \otimes \frac{\partial}{\partial y^j},$$
$$\sum_{i=1}^{m} dx^i \otimes \frac{\partial}{\partial x^i} + \sum_{i,k,l=1}^{m}\sum_{|\gamma|\leqslant K} \nabla_{ik\gamma}^{l} x^\gamma dx^i \otimes V_l^k)(u, v_o) = 0$$

*for any $K \in \mathbb{N}$, any $u \in (V^F\mathbb{R}^{m,n})_0$, any $\Gamma_{i\alpha\beta}^{j}$ and any $\nabla_{ik\gamma}^{l}$ with*

$$\nabla_{ik(0)}^{l} = -\nabla_{ki(0)}^{l}$$

*for $i,j,k,l,\alpha,\beta,\gamma$ as indicated, where $v_o = \frac{\partial}{\partial x^1}(0) \in T_0\mathbb{R}^m$ and $V_l^k$ is the vertical vector field on $T\mathbb{R}^m$ such that $V_l^k(v) = ((x^k - x_o^k)\frac{\partial}{\partial x^l})^C(v)$ for any $v \in T_{x_o}\mathbb{R}^m$, $x_o \in \mathbb{R}^m$ and $X^C$ is the complete lifting (flow prolongation) of $X \in \mathcal{X}(M)$ to $TM$. Then $D = 0$.*

PROOF. The principal part of the lemma follows from a corollary of non-linear Peetre theorem (Corollary 19.8 in [**5**]). From the existence of normal coordinate systems of classical linear connections and from the naturality of $D$ it follows that we can restrict to $\nabla_{ik(0)}^{l} = -\nabla_{ki(0)}^{l}$. (We recall that in normal coordinates with centrum in 0 we have $\nabla_{ik}^{l}(0) + \nabla_{ki}^{l}(0) = 0$.) Finally, that we can restrict to $v_o = \frac{\partial}{\partial x^1}(0) \in T_0\mathbb{R}^m$ follows from the fact that any $v_o \in T_0\mathbb{R}^m \setminus \{0\}$ can be transformed into $\frac{\partial}{\partial x^1}(0)$ by a linear isomorphism, from the naturality of $D$ and from the fact that the composition of a normal coordinate system with a linear isomorphism is a normal coordinate system. $\square$

LEMMA 4. *Suppose that*

$$D(\sum_{i=1}^{m} dx^i \otimes \frac{\partial}{\partial x^i} + y^\beta dx^{i_o} \otimes \frac{\partial}{\partial y^{j_o}}, \sum_{i=1}^{m} dx^i \otimes \frac{\partial}{\partial x^i})(u, v_o) = 0$$

*and*

$$D(\sum_{i=1}^{m} dx^i \otimes \frac{\partial}{\partial x^i}, \sum_{i=1}^{m} dx^i \otimes \frac{\partial}{\partial x^i} + dx^{\overline{i}_o} \otimes V_{l_o}^{k_o})(u, v_o) = 0$$

*for any $u \in (V^F\mathbb{R}^{m,n})_0$, any $n$-tuple $\beta$ and any $i_o, \overline{i}_o, k_o, l_o = 1, ..., m$ and $j_o = 1, ..., n$ with $\overline{i}_o < k_o$, where $v_o = \frac{\partial}{\partial x^1}(0) \in T_0\mathbb{R}^m$. Then $D = 0$.*

PROOF. Using the invariance of $D$ with respect to the base homotheties $tid_{\mathbb{R}^m} \times id_{\mathbb{R}^n}$ for $t > 0$ we get the homogeneity condition

$$D(\sum_{i=1}^{m} dx^i \otimes \frac{\partial}{\partial x^i} + \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{|\alpha|+|\beta|\leqslant K} \Gamma^j_{i\alpha\beta} t^{|\alpha|+1} x^\alpha y^\beta dx^i \otimes \frac{\partial}{\partial y^j},$$
$$\sum_{i=1}^{m} dx^i \otimes \frac{\partial}{\partial x^i} + \sum_{i,k,l=1}^{m}\sum_{|\gamma|\leqslant K} \nabla^l_{ik\gamma} t^{|\gamma|+1} x^\gamma dx^i \otimes V^k_l)(u, v_o) =$$
$$tD(\sum_{i=1}^{m} dx^i \otimes \frac{\partial}{\partial x^i} + \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{|\alpha|+|\beta|\leqslant K} \Gamma^j_{i\alpha\beta} x^\alpha y^\beta dx^i \otimes \frac{\partial}{\partial y^j},$$
$$\sum_{i=1}^{m} dx^i \otimes \frac{\partial}{\partial x^i} + \sum_{i,k,l=1}^{m}\sum_{|\gamma|\leqslant K} \nabla^l_{ik\gamma} x^\gamma dx^i \otimes V^k_l)(u, v_o).$$

By the homogeneous function theorem from [**5**], this type of homogeneity gives that

$$D(\sum_{i=1}^{m} dx^i \otimes \frac{\partial}{\partial x^i} + \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{|\alpha|+|\beta|\leqslant K} \Gamma^j_{i\alpha\beta} x^\alpha y^\beta dx^i \otimes \frac{\partial}{\partial y^j},$$
$$\sum_{i=1}^{m} dx^i \otimes \frac{\partial}{\partial x^i} + \sum_{i,k,l=1}^{m}\sum_{|\gamma|\leqslant K} \nabla^l_{ik\gamma} x^\gamma dx^i \otimes V^k_l)(u, v_o)$$

depends linearly on the $\Gamma^j_{i(0)\beta}$ and $\nabla^l_{ik(0)}$ and it is independent of the $\Gamma^j_{i\alpha\beta}$ and $\nabla^l_{ik\gamma}$ for $|\alpha| > 0$ and $|\gamma| > 0$. (In particular, we have

$$D(\sum_{i=1}^{m} dx^i \otimes \frac{\partial}{\partial x^i}, \sum_{i=1}^{m} dx^i \otimes \frac{\partial}{\partial x^i})(u, v_o) = 0 \tag{**}$$

what we will apply in the proof of Lemma 7.) So, the assumptions of the lemma imply the assumptions of Lemma 3, which completes the proof. □

LEMMA 5. *Suppose that*

$$D(\sum_{i=1}^{m} dx^i \otimes \frac{\partial}{\partial x^i} + dx^{i_o} \otimes Y, \sum_{i=1}^{m} dx^i \otimes \frac{\partial}{\partial x^i})(u, v_o) = 0$$

*and*

$$D(\sum_{i=1}^{m} dx^i \otimes \frac{\partial}{\partial x^i}, \sum_{i=1}^{m} dx^i \otimes \frac{\partial}{\partial x^i} + dx^{\bar{i}_o} \otimes V^{k_o}_{l_o})(u, v_o) = 0$$

*for any* $u \in (V^F\mathbb{R}^{m,n})_0$, *any vertical vector field* $Y$ *on* $\mathbb{R}^{m,n}$ *and any* $i_o, \bar{i}_o, k_o, l_o = 1, ..., m$ *with* $\bar{i}_o < k_o$, *where* $v_o = \frac{\partial}{\partial x^1}(0) \in T_0\mathbb{R}^m$. *Then* $D = 0$.

PROOF. The assumptions of this lemma imply the assumptions of Lemma 4. □

LEMMA 6. *Suppose that*

$$D(\sum_{i=1}^{m} dx^i \otimes \frac{\partial}{\partial x^i} + dx^{i_o} \otimes \frac{\partial}{\partial y^1}, \sum_{i=1}^{m} dx^i \otimes \frac{\partial}{\partial x^i})(u, v_o) = 0$$

*and*

$$D(\sum_{i=1}^{m} dx^i \otimes \frac{\partial}{\partial x^i}, \sum_{i=1}^{m} dx^i \otimes \frac{\partial}{\partial x^i} + dx^{\bar{i}_o} \otimes V_{l_o}^{k_o})(u, v_o) = 0$$

*for any* $u \in (V^F\mathbb{R}^{m,n})_0$, *any* $i_o, \bar{i}_o, k_o, l_o = 1, ..., m$ *with* $\bar{i}_o < k_o$, *where* $v_o = \frac{\partial}{\partial x^1}(0) \in T_0\mathbb{R}^m$. *Then* $D = 0$.

Proof. By the Frobenius theorem, any non-vanishing vertical vector field $Y$ on $\mathbb{R}^{m,n}$ can be transformed locally into $\frac{\partial}{\partial y^1}$ by a fibered isomorphism of the form $id_{\mathbb{R}^m} \times \psi$. Then (by the naturality of $D$ with respect to fibered isomorphisms of the form $id_{\mathbb{R}^m} \times \psi$) the assumptions of this lemma imply the assumptions of Lemma 5, which completes the proof. □

Lemma 7. *Suppose that*

$$D(\sum_{i=1}^{m} dx^i \otimes \frac{\partial}{\partial x^i}, \sum_{i=1}^{m} dx^i \otimes \frac{\partial}{\partial x^i} + dx^{\bar{i}_o} \otimes V_{l_o}^{k_o})(u, v_o) = 0$$

*for any* $u \in (V^F\mathbb{R}^{m,n})_0$, *any* $\bar{i}_o, k_o, l_o = 1, ..., m$ *with* $\bar{i}_o < k_o$, *where* $v_o = \frac{\partial}{\partial x^1}(0) \in T_0\mathbb{R}^m$. *Then* $D = 0$.

Proof. Using the naturality condition of $D$ with respect to the $\mathcal{FM}_{m,n}$-isomorphism $(x^1, ..., x^m,\ y^1 + x^{i_o}, y^2, ..., y^n)$ we see that from (**) it follows the first equality of the assumptions of Lemma 6. Then Lemma 6 completes the proof. □

Lemma 8. *Suppose that*

$$D(\sum_{i=1}^{m} dx^i \otimes \frac{\partial}{\partial x^i}, \sum_{i=1}^{m} dx^i \otimes \frac{\partial}{\partial x^i} + dx^{\bar{i}_o} \otimes V_{k_o}^{k_o})(u, v_o) = 0$$

*for any* $u \in (V^F\mathbb{R}^{m,n})_0$, *any* $\bar{i}_o, k_o = 1, ..., m$ *with* $\bar{i}_o < k_o$, *where* $v_o = \frac{\partial}{\partial x^1}(0) \in T_0\mathbb{R}^m$. *Then* $D = 0$.

Proof. We prove that from the assumption of this lemma it follows the assumption of Lemma 7. Let $\bar{i}_o, k_o, l_o = 1, \dots, m$ be such that $\bar{i}_o < k_o$ and $k_o \neq l_o$. Then $k_o > 1$. Using the naturality of $D$ with respect to the $\mathcal{FM}_{m,n}$-diffeomorphism

$$(x^1, ..., tx^{k_o}, ..., x^m, y^1, ..., y^n)$$

(only $k_o$-position is exceptional) for $t \neq 0$ we get the condition

$$D(\sum_{i=1}^{m} dx^i \otimes \frac{\partial}{\partial x^i}, \sum_{i=1}^{m} dx^i \otimes \frac{\partial}{\partial x^i} + t dx^{\bar{i}_o} \otimes V_{l_o}^{k_o})(u, v_o) =$$
$$= D(\sum_{i=1}^{m} dx^i \otimes \frac{\partial}{\partial x^i}, \sum_{i=1}^{m} dx^i \otimes \frac{\partial}{\partial x^i} + dx^{\bar{i}_o} \otimes V_{l_o}^{k_o})(u, v_o)\ .$$

Then putting $t \to 0$ and applying (**) we get

$$D(\sum_{i=1}^{m} dx^i \otimes \frac{\partial}{\partial x^i}, \sum_{i=1}^{m} dx^i \otimes \frac{\partial}{\partial x^i} + dx^{\bar{i}_o} \otimes V_{l_o}^{k_o})(u, v_o) = 0$$

which is nothing else but the assumption of Lemma 7. □

LEMMA 9. *Suppose that*

$$D(\sum_{i=1}^m dx^i \otimes \frac{\partial}{\partial x^i}, \sum_{i=1}^m dx^i \otimes \frac{\partial}{\partial x^i} + dx^1 \otimes V_2^2)(u, v_o) = 0$$

*for any* $u \in (V^F \mathbb{R}^{m,n})_0$, *where* $v_o = \frac{\partial}{\partial x^1}(0) \in T_0\mathbb{R}^m$. *Then* $D = 0$.

PROOF. Let $\bar{i}_o, k_o = 1, ..., m$ be such that $\bar{i}_o < k_o$. Using the naturality of $D$ with respect to the $\mathcal{FM}_{m,n}$-morphism replacing $x^2$ by $x^{k_o}$ and vice-versa and next the naturality of $D$ with respect to the $\mathcal{FM}_{m,n}$-morphism $(x^1 - x^{\bar{i}^o}, x^2, ..., x^m, y^1, ..., y^n)$ and the last sentence before formula (**), we obtain that from the assumption of this lemma it follows the assumption of Lemma 8. □

**We are now in position to prove Theorem 2.** Using $D$, we can define a vector field $C$ on $F\mathbb{R}^n$ by

$$C(u) = D(\sum_{i=1}^m dx^i \otimes \frac{\partial}{\partial x^i}, \sum_{i=1}^m dx^i \otimes \frac{\partial}{\partial x^i} + dx^1 \otimes V_2^2)(u, v_o)$$

for $u \in F\mathbb{R}^n \tilde{=} (V^F\mathbb{R}^{m,n})_0$, $0 \in \mathbb{R}^m$. It is easily seen (using the naturality of $D$) that $C$ is invariant with respect to local diffeomorphisms $\mathbb{R}^n \to \mathbb{R}^n$. So, $C$ corresponds to an $\mathcal{M}f_n$-natural vector field $C$ on $F$. Further, one verifies directly

$$D(\sum_{i=1}^m dx^i \otimes \frac{\partial}{\partial x^i}, \sum_{i=1}^m dx^i \otimes \frac{\partial}{\partial x^i} + dx^1 \otimes V_2^2)(u, v_o) =$$
$$= (\sum_{i=1}^m dx^i \otimes \frac{\partial}{\partial x^i}, \sum_{i=1}^m dx^i \otimes \frac{\partial}{\partial x^i} + dx^1 \otimes V_2^2)^{F,C}(u, v_o) .$$

Then $D = (\ ,\ )^{F,C}$ because of Lemma 9. The proof of Theorem 2 is complete.

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