**DIFFERENTIAL GEOMETRY AND ITS APPLICATIONS**
Proc. Conf. Prague, August 30 – September 3, 2004
Charles University, Prague (Czech Republic), 2005, 571 – 579

# Non-holonomic mechanical systems and variational principle

Lenka Czudková, Jitka Janová and Jana Musilová

Abstract. Within the framework of mechanics we often meet systems subjected to various types of constraints. Naturally, it is of interest to derive corresponding equations of motion and then to deal with a question of their variationality. While both equations of motion and variational theory concerning the holonomic systems are well-known, non-holonomic situations are still studied by many authors.

Recently, the geometrical theory of mechanical systems with non-holonomic constraints on jet manifolds was developed by Krupková and subsequently a new concept of *constraint variationality* for obtained equations of motion was proposed by Krupková and Musilová. Applying the results of these theories to physical examples we discuss some variational aspects of non-holonomic mechanical systems (equivalence properties of constraint Cartan forms, formulation of constraint variational principle) and compare them with an unconstrained case.

## 1. Introduction

An important family of mechanical systems constitute variational ones, i.e. systems whose equations of motion can be derived from a Lagrangian. However, only the variational theory of unconstrained mechanical systems and those subjected to the holonomic constraints is satisfactorily elaborated from both physical and mathematical point of view. On the other hand, in physics we also meet more general types of constrained systems and thus there arises a necessity to study them in general. That is a subject of interest of many authors (see References).

We adopt the geometrical theory of mechanical systems subjected to non-holonomic constraints developed by Krupková (see [**3**]), the concept of variationality of corresponding equations of motion proposed later by Krupková and Musilová (see [**8**], [**9**]) and a new concept of the non-holonomic variational principle formulated by Krupková and Swaczyna (see [**7**], [**18**]). Our main purpose is to apply results of these theories (briefly summarized in Section 2) to concrete physical examples and through them to demonstrate some new variational aspects that arise in consequence of a presence of non-holonomic constraints (Section 3). We there consider typical initially non-variational unconstrained systems subjected to such constraints to turn into constrained variational ones. By these situations obstructions for the use of the

2000 *Mathematics Subject Classification.* 49S05, 37J60, 70F25.
*Key words and phrases.* Fibred manifold, dynamical form, first-order mechanical system, constraint manifold, non-holonomic constraints, (constrained) variational system, (constraint) Cartan form.
This paper is in final form and no version of it will be published elswhere.

variational principle analogous to the unconstrained one (see e.g. [**1**]) are pointed out.

Throughout this paper the standard notation of geometrical objects is used. Let $\pi : Y \to X$ be a fibred manifold, $\dim Y = m + 1$, $\dim X = 1$, $\pi_1 : J^1Y \to X$ and $\pi_2 : J^2Y \to X$ be its first and second jet prolongation, and $\pi_{r,s} : J^rY \to J^sY$, $0 \leqslant r, s \leqslant 2$, $s < r$, $J^0Y = Y$, be canonical projections. If we denote by $(V, \psi)$, $\psi = (t, q^\sigma)$, $1 \leqslant \sigma \leqslant m$, a fibred chart on $Y$, then there arise associated fibred charts $(V_1, \psi_1)$, $V_1 = \pi_{1,0}^{-1}(V)$, $\psi_1 = (t, q^\sigma, \dot{q}^\sigma)$ and $(V_2, \psi_2)$, $V_2 = \pi_{2,0}^{-1}(V)$, $\psi_1 = (t, q^\sigma, \dot{q}^\sigma, \ddot{q}^\sigma)$ on $J^1Y$ and on $J^2Y$. A section $\delta : I \to J^rY$ of $\pi_r$, $I \subset X$ is an open set, $1 \leqslant r \leqslant 2$, is called *holonomic* if there is a section $\gamma$ of $\pi$ such that $\delta = J^r\gamma$.

A vector field $\xi$ on $J^sY$, $0 \leqslant s \leqslant 2$, is called *$\pi_s$-projectable* if there exists a vector field $\xi_0$ on $X$ such that $T\pi_s\xi = \xi_0 \circ \pi_s$, and is called *$\pi_s$-vertical* if $T\pi_s \, \xi = 0$. By analogy, a $\pi_{r,s}$-projectable vector field and a $\pi_{r,s}$-vertical vector field is defined. A form $\eta$ on $J^sY$ is called *$\pi_s$-horizontal* if $i_\xi \eta = 0$ for every $\pi_s$-vertical vector field $\xi$ on $J^sY$, and is called *contact* if $J^s\gamma^*\eta = 0$, $s > 0$, for every section $\gamma$ of $\pi$. A $\pi_{r,s}$-horizontal form is defined analogously. We will use a basis $(\mathrm{d}t, \omega^\sigma, \mathrm{d}\dot{q}^\sigma)$ of 1-forms on $J^1Y$ and a basis $(\mathrm{d}t, \omega^\sigma, \dot{\omega}^\sigma, \mathrm{d}\ddot{q}^\sigma)$ of 1-forms on $J^2Y$ where $\omega^\sigma = \mathrm{d}q^\sigma - \dot{q}^\sigma \mathrm{d}t$, $\dot{\omega}^\sigma = \mathrm{d}\dot{q}^\sigma - \ddot{q}^\sigma \mathrm{d}t$ are contact 1-forms. Recall that every $k$-form $\eta$ on $J^sY$ can be uniquely decomposed as follows: $\pi_{s+1,s}^*\eta = p_{k-1}\eta + p_k\eta$ where $p_{k-1}\eta$ and $p_k\eta$ are called $(k-1)$-*contact component* and *$k$-contact component* of $\eta$, respectively. A $k$-form $\eta$ is called *$k$-contact* if $p_{k-1}\eta = 0$, and is called $(k-1)$-contact if $p_k\eta = 0$. For $k = 1$ we have $p_{k-1}\eta = p_0\eta = h\eta$ the *horizontal component* of $\eta$.

A *distribution* $\mathcal{D}$ on $J^sY$ is defined as a mapping assigning to every point $x \in J^sY$ a vector subspace $\mathcal{D}(x) \subset T_xJ^sY$. A distribution is generated either by a system of local vector fields $\xi_\iota$ on $J^sY$, $\iota \in \mathcal{I}$, or by its *annihilators*, 1-forms $\eta_\kappa$ on $J^sY$, $\kappa \in \mathcal{K}$, such that $i_{\xi_\iota}\eta_\kappa = 0$ for every $\iota \in \mathcal{I}$, $\kappa \in \mathcal{K}$ where $\mathcal{I}, \mathcal{K}$ are sets of indices. We write $\mathcal{D} = \mathrm{span}\,\{\, \xi_\iota \mid \iota \in \mathcal{I} \,\}$ and $\mathcal{D}^0 = \mathrm{span}\,\{\, \eta_\kappa \mid \kappa \in \mathcal{K} \,\}$. A section $\delta$ of $\pi_s$ is called an *integral section* of a distribution $\mathcal{D}$ if $\delta^*\eta = 0$ for every $\eta \in \mathcal{D}^0$.

## 2. Basic geometrical concepts

A geometrical theory of mechanical systems developed by Krupková (for details and proofs see [**3**]) is based on a concept of *dynamical form* $E$, defined as a 1-contact $\pi_{2,0}$-horizontal 2-form on $J^2Y$. Such a form is in every fibred chart expressed as follows:

$$E = E_\sigma\,(t, q^\rho, \dot{q}^\rho, \ddot{q}^\rho)\,\omega^\sigma \wedge \mathrm{d}t\,, \qquad 1 \leqslant \sigma, \rho \leqslant m\,.$$

A section $\gamma$ of $\pi$ is called a *path* of dynamical form $E$ if

$$E \circ J^2\gamma = 0\,.$$

With regard to physical applications, only dynamical forms affine in the second derivatives (accelerations) are considered, i.e.

$$E_\sigma\,(t, q^\rho, \dot{q}^\rho, \ddot{q}^\rho) = A_\sigma\,(t, q^\rho, \dot{q}^\rho) + B_{\sigma\nu}\,(t, q^\rho, \dot{q}^\rho)\,\ddot{q}^\nu\,, \qquad 1 \leqslant \sigma, \rho, \nu \leqslant m\,.$$

Equations of paths then take the form

$$A_\sigma + B_{\sigma\nu}\ddot{q}^\nu = 0 \quad \text{along} \quad J^2\gamma$$

and are called (*unconstrained*) *equations of motion.*

Every dynamical form $E$ is associated with a class of 2-forms $\alpha$ on $J^1Y$ satisfying the condition $p_1\alpha = E$. It holds

$$\alpha = A_\sigma\, \omega^\sigma \wedge \mathrm{d}t + B_{\sigma\nu}\, \omega^\sigma \wedge \mathrm{d}\dot{q}^\nu + F_{\sigma\nu}\omega^\sigma \wedge \omega^\nu$$

where $F_{\sigma\nu} = -F_{\nu\sigma}$ are some free functions on $J^1Y$. The equivalence class $[\alpha]$ with equivalence relation $\alpha_1 \sim \alpha_2$ if $\alpha_1 - \alpha_2$ is a 2-contact 2-form is called a *Lepagean class of* $E$ or a *first-order mechanical system associated to* $E$. Thus, a path of a dynamical form $E$ will be also called a *path of the mechanical system* $[\alpha]$ *associated to* $E$.

**Proposition 1.** *A section* $\gamma$ *is a path of a mechanical system* $[\alpha]$ *iff* $J^1\gamma^* i_\xi \alpha = 0$ *for every* $\pi_1$*-vertical vector field* $\xi$ *on* $J^1Y$.

A dynamical form $E$ is called (*locally*) *variational* if in a neighbourhood of every point $x \in J^1Y$ there exists a Lagrangian $\lambda = L\,(t, q^\sigma, \dot{q}^\sigma)\,\mathrm{d}t$ such that

$$E_\sigma = \frac{\partial L}{\partial q^\sigma} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}^\sigma}\,.$$

The equations of motion are then called *Euler–Lagrange equations.*

**Theorem 1.** *A dynamical form* $E$ *is (locally) variational iff the corresponding mechanical system* $[\alpha]$ *contains a closed representative. Such a representative is unique.*

**Remark 1.** The closed representative $\alpha_E \in [\alpha]$ in a variational mechanical system can be expressed e.g. by means of the *Cartan form* $\theta = L\mathrm{d}t + \frac{\partial L}{\partial \dot{q}^\sigma}\omega^\sigma$ as $\alpha_E = \mathrm{d}\theta$.

**Remark 2.** A mechanical system is variational iff functions $A_\sigma$ and $B_{\sigma\nu}$ satisfy well-known Helmholtz conditions (for details see [**4**]). Recall that in such a case a section $\gamma$ of $\pi$ is a path of mechanical system $[\alpha]$ on an interval $[t_1, t_2] \in X$ iff so-called *variational integral*

$$\int_{t_1}^{t_2} J^1\gamma^*\lambda = \int_{t_1}^{t_2} J^1\gamma^*\theta = \int_{t_1}^{t_2} J^1\gamma^* L\mathrm{d}t$$

is stationary.

Now, let us turn to the geometrical theory of systems with non-holonomic constraints (see again [**3**]). First, by a *constraint manifold* $Q$ we mean a submanifold of $J^1Y$ fibred over $Y$, $\dim Q = 2m + 1 - k\,,\ 1 \leqslant k \leqslant m - 1$, (locally) defined by a system of *non-holonomic constraints* as follows:

$$f^i\,(t, q^\sigma, \dot{q}^\sigma) = 0\,, \qquad \operatorname{rank}\left(\frac{\partial f^i}{\partial \dot{q}^\sigma}\right) = k\,, \qquad 1 \leqslant i \leqslant k\,.$$

This system can be locally solved and rewritten into a more frequent *normal form*

$$\dot{q}^{m-k+i} - g^i\left(t, q^\sigma, \dot{q}^l\right) = 0\,, \qquad 1 \leqslant l \leqslant m - k\,.$$

Having still on mind physical applications, only sections $\gamma$ of $\pi$ satisfying the condition $f^i \circ J^1\gamma = 0$, called *Q-admissible sections* (see [**7**], [**18**]), should be considered as possible paths. Our aim is to formulate necessary and sufficient condition for the section $\gamma$ of $\pi$ to be $Q$-admissible. For this purpose, let $U_\iota$ be an open set (in $J^1Y$) satisfying the condition $U_\iota \cap Q \neq \emptyset$ and on $U_\iota$ put

$$\varphi^i = f^i\mathrm{d}t + \frac{\partial f^i}{\partial \dot{q}^\sigma}\,\omega^\sigma\,, \qquad 1 \leqslant i \leqslant k\,.$$

Forms $\varphi^i$ are called (*local*) *constraint* 1-*forms on* $U_\iota$ and together with the forms $\mathrm{d}f^i$ they generate the *constraint distribution* $\mathcal{C}_{U_\iota}$ on $U_\iota$:

$$\mathcal{C}^0_{U_\iota} = \operatorname{span}\left\{\varphi^i\,,\mathrm{d}f^i \mid 1 \leqslant i \leqslant k\right\}\,.$$

Finally, denote

$$\varphi^i = \iota^*\varphi^i$$

where $\iota : Q \to J^1Y,\ \iota\,(t, q^\sigma, \dot q^\sigma) = \left(t, q^\sigma, \dot q^l, g^i\,(t, q^\nu, \dot q^r)\right)$, is canonical embedding of $Q$ into $J^1Y$.

**Proposition 2.**
*(a) Denote* $\mathcal{C}^0 = \operatorname{span}\left\{\varphi^i\right\} = \operatorname{span}\left\{\iota^*\varphi^i\right\}$ *where* $\varphi^i$ *run over the set of all constraint* 1*-forms on the open sets* $U_\iota$ *covering* $Q$*. Then* $\mathcal{C}$ *is a distribution of corank* $k$ *(with respect to* $Q$*) on* $Q$*.*
*(b) A section* $\gamma$ *of* $\pi$ *satisfies the condition* $f^i \circ J^1\gamma = 0$ *(i.e.* $\gamma$ *is a* $Q$*-admissible section) iff* $J^1\gamma$ *is an integral section of* $\mathcal{C}$*.*

Distribution $\mathcal{C}$ on $Q$ is called *canonical distribution.* It holds

$$\varphi^i = -\frac{\partial g^i}{\partial \dot q^l}\,\omega^l + \mathrm{d}q^{m-k+i} - g^i\mathrm{d}t\,,$$

i.e. we can introduce a new basis $\left(\mathrm{d}t, \omega^l, \varphi^i, \mathrm{d}\dot q^l\right)$ of 1-forms on $Q$.

Denote $\alpha_Q = \iota^*\alpha,\ \alpha \in [\alpha]$, and put $\iota^*\alpha_1 \sim \iota^*\alpha_2$ if $\iota^*\alpha_1 - \iota^*\alpha_2 = \bar F_{ls}\,\omega^l \wedge \omega^s + \varphi^i \wedge \chi_i$ where $\bar F_{ls} = -\bar F_{sl}$ are some free functions on $Q$ and $\chi_i$ is a 1-form on $Q$. The equivalence class $[\alpha_Q] = [\iota^*\alpha]$ is called a *constrained system on* $Q$ *related to the mechanical system* $[\alpha]$. A direct calculation gives

$$\alpha_Q = \bar A_l\,\omega^l \wedge \mathrm{d}t + \bar B_{ls}\,\omega^l \wedge \mathrm{d}\dot q^s + \bar F_{ls}\,\omega^l \wedge \omega^s + \varphi^i \wedge \chi_i$$

where

$$\bar A_l =$$

$$\Big(A_l + \sum_{j=1}^{k} A_{m-k+j}\frac{\partial g^j}{\partial \dot q^l} + \sum_{i=1}^{k}\big(B_{l,m-k+i} + \sum_{j=1}^{k} B_{m-k+j,m-k+i}\frac{\partial g^j}{\partial \dot q^l}\big)\big(\frac{\partial g^i}{\partial t} + \frac{\partial g^i}{\partial q^\sigma}\dot q^\sigma\big)\Big) \circ \iota$$

$$\bar B_{ls} = \left(B_{ls} + \sum_{i=1}^{k}\left(B_{l,m-k+i}\frac{\partial g^i}{\partial \dot q^s} + B_{m-k+i,s}\frac{\partial g^i}{\partial \dot q^l}\right) + B_{m-k+j,m-k+i}\frac{\partial g^j}{\partial \dot q^l}\frac{\partial g^i}{\partial \dot q^s}\right) \circ \iota\,.$$

A path of a constrained system is defined in analogy to a property of an unconstrained case (cf. Proposition 1):

**Definition 1.** A section $\gamma$ is called a *path of a constrained system* $[\alpha_Q]$ if

$$f^i \circ J^1\gamma = 0 \quad (\text{i.e. } \gamma \text{ is a } Q-\text{admissible section})$$

together with one of the equivalent conditions

$$\begin{aligned} J^1\gamma^* i_{\xi_{\mathcal{C}}}\alpha_Q &= 0 \qquad \text{for every } \pi_1-\text{vertical vector field } \xi_{\mathcal{C}} \text{ belonging to } \mathcal{C}\,,\\ \bar A_l + \bar B_{ls}\ddot q^{\,s} &= 0 \qquad \text{along } J^2\gamma\,. \end{aligned}$$

The equations $f^i \circ J^1\gamma = 0,\ \bar A_l + \bar B_{ls}\ddot q^{\,s} = 0$ are called *reduced equations of motion.* They represent a system of $m$ ordinary differential equations for $q^\sigma \circ \gamma\,(t)$.

**Remark 3.** The above presented geometrical approach is equivalent to the physical one for a *constraint force* of the (local) form on $U_\iota$

$$\Phi_{U_\iota} = \mu^i \frac{\partial f^i}{\partial \dot{q}^\sigma} \omega^\sigma \wedge \mathrm{d}t\,.$$

Such a force is known as *Chetaev force* and functions $\mu^i$ on $J^1Y$ as *Lagrange multipliers* (see [**3**]). By means of Chetaev force one obtains so-called *deformed equations of motion*

$$\begin{aligned} f^i \circ J^1\gamma &= 0\,, \\ A_\sigma + B_{\sigma\nu}\ddot{q}^\nu &= \mu^i \frac{\partial f^i}{\partial \dot{q}^\sigma} \quad \text{along} \quad J^2\gamma\,. \end{aligned}$$

They represent a system of $(m+k)$ ordinary differential equations for $q^\sigma \circ \gamma\,(t)$ and $\mu^i \circ J^1\gamma\,(t)$ with the same solutions for $q^\sigma \circ \gamma\,(t)$ like those of reduced equations.

Note that the theory does not cover all possible situations because only $Q$-admissible sections satisfying corresponding reduced/deformed equations of motion are considered here. In other words, from the physical point of view there exist trajectories guaranteed by constraint forces that are not of Chetaev type.

A concept of variationality of a constrained system is defined as follows (see [**8**], [**9**], cf. Theorem 1):

**Definition 2.** A constrained system $[\alpha_Q]$ is called *variational* if it contains a closed representative.

**Remark 4.**
(a) Contrary to an unconstrained case, a closed representative in a constrained variational system need not be unique.
(b) A constrained system arising from the variational one is variational (in the sense of Definition 2). The converse is not true in general.
(c) A constrained system is variational iff so-called *constraint Helmholtz conditions* hold (see [**8**], [**9**]).

**Proposition 3.** *Let $[\alpha_Q]$ be a variational constrained system.*
*(a) There exists a representative $\alpha_Q$ and a local 1-form $\bar{\theta} = \bar{L} + \frac{\partial \bar{L}}{\partial \dot{q}^l}\,\omega^l + \bar{L}_{m-k+i}\,\varphi^i$ such that $\alpha_Q = \mathrm{d}\bar{\theta}$ where $\bar{L}$, $\bar{L}_{m-k+i}$, $1 \leqslant i \leqslant k$, are functions on $Q$.*
*(b) Let $\varepsilon_l$ be the operator defined as follows:*

$$\varepsilon_l = \frac{\partial_c}{\partial q^l} - \frac{\mathrm{d}'_c}{\mathrm{d}t}\frac{\partial}{\partial \dot{q}^l} - \ddot{q}^s \frac{\partial^2}{\partial \dot{q}^l \partial \dot{q}^s}$$

*where*

$$\frac{\partial_c}{\partial q^l} = \frac{\partial}{\partial q^l} + \frac{\partial g^j}{\partial \dot{q}^l}\frac{\partial}{\partial q^{m-k+j}}\,, \qquad \frac{\mathrm{d}'_c}{\mathrm{d}t} = \frac{\partial}{\partial t} + \dot{q}^l \frac{\partial}{\partial q^l} + g^j \frac{\partial}{\partial q^{m-k+j}}\,.$$

*Then the reduced equations of motion are of the form*

$$f^i \circ J^1\gamma = 0\,, \qquad \varepsilon_l\left(\bar{L}\right) - \bar{L}_{m-k+i}\,\varepsilon_l\left(g^i\right) = 0\,.$$

**Definition 3.** A 1-form $\bar{\theta} = \bar{L}\mathrm{d}t + \frac{\partial \bar{L}}{\partial \dot{q}^l}\,\omega^l + \bar{L}_{m-k+i}\,\varphi^i$ is called a *constraint Cartan form*. The expressions $\varepsilon_l\left(\bar{L}\right) - \bar{L}_{m-k+i}\,\varepsilon_l\left(g^i\right)$ are called *constraint Euler–Lagrange expressions* (cf. [**6**], [**7**], [**9**], [**18**]).

In order to answer a question of variationality of a constrained mechanical system we can use one of the following equivalent methods:

- to verify constraint Helmholtz conditions for functions $\bar{A}_l$, $\bar{B}_{ls}$ (see [**8**], [**9**]),
- to find out if there exists a constraint Cartan form that leads to corresponding reduced equations of motion (see Proposition 3).

Note that constraint Cartan forms are given by $(k+1)$ functions $\bar{L}$, $\bar{L}_{m-k+i}$.

**Definition 4.** A 1-form $\bar{\tau} = \bar{T}\mathrm{d}t + \frac{\partial \bar{T}}{\partial \dot{q}^l}\,\omega^l + \bar{T}_{m-k+i}\,\varphi^i$ is called a *trivial constraint Cartan form* if $\varepsilon_l\left(\bar{T}\right) - \bar{T}_{m-k+i}\,\varepsilon_l\left(g^i\right) \equiv 0$.

**Remark 5.**
While all trivial Cartan forms of an unconstrained variational system are just the exterior derivatives of functions $h\,(t, q^\sigma)$, trivial constraint Cartan forms are given by all closed forms of the type $\rho = \bar{F}_{ls}\,\omega^l \wedge \omega^s + \varphi^i \wedge \chi_i$ as $\mathrm{d}\bar{\tau} = \rho$ (cf. Remark 4(a)).

## 3. Constraint Cartan forms in physical examples

The main purpose of this paper is to study some properties of constraint Cartan forms. In this section we present two concrete examples very similar from the physical point of view but with quite different properties of corresponding constraint Cartan forms.

Consider a point particle of mass $m$ performing a two-dimensional motion under Stokes force. Such a motion is studied on the fibred manifold $\pi : \left(\mathbf{R} \times \mathbf{R}^2\right) \to \mathbf{R}$ and its prolongations. If we denote by $\psi = (t, x, y)$ fibred coordinates on $Y$, then $F_{\text{Stokes}} = -\beta\,(\dot{x}, \dot{y})$ where $\beta > 0$ is a constant. A non-holonomic constraint will be given by the conservation of the mechanical energy.

In general, for constrained systems arising from non-variational unconstrained ones both possibilities can occur – the constrained system is variational or not, depending not only on the initially given unconstrained system but also on the constraints. Our examples are chosen in such a way that the non-holonomic constraint leads to the variational constrained system. Its variationality will be verified by finding a constraint Cartan form that is now of the type

$$\bar{\theta} = \bar{L}\mathrm{d}t + \frac{\partial \bar{L}}{\partial \dot{x}}\,\omega^1 + \bar{L}_2\,\varphi^1 \,.$$

**Example 1.**
Assume there is only Stokes force acting on the particle. Unconstrained equations of motion

$$m\ddot{x} + \beta\dot{x} = 0\,, \qquad m\ddot{y} + \beta\dot{y} = 0$$

are non-variational. The non-holonomic constraint representing the conservation of the mechanical energy takes the form

$$\frac{1}{2}m\,(\dot{x}^2 + \dot{y}^2) = E_0 = const\,.$$

Reduced equations of motion on $Q_+ = \{a \mid a \in Q,\, \dot{y}(a) > 0\}$

$$g^1 = \dot{y} = \sqrt{\frac{2E_0}{m} - \dot{x}^2}\,, \qquad m\ddot{x}\left(1 + \frac{\dot{x}^2}{(g^1)^2}\right) = 0$$

are variational. There exist three different types of corresponding constraint Cartan forms:

(a) $\bar{L} = 0$, $\bar{L}_2 \neq 0$, e.g.

$$\bar{\theta}_{(a)} = -mg^1\varphi^1 + \mathrm{d}h$$

where $h(t,x,y)$ is a function on $Q_+$.

(b) $\bar{L} \neq 0$, $\bar{L}_2 = 0$, e.g.

$$\bar{\theta}_{(b)} = \left(-E_0 \ln(g^1)^2 + \sqrt{\frac{E_0 m}{2}}\,\dot{x}\,\ln\frac{\sqrt{\frac{2E_0}{m}} - \dot{x}}{\sqrt{\frac{2E_0}{m}} + \dot{x}}\right)\mathrm{d}t + \left(\sqrt{\frac{E_0 m}{2}}\ln\frac{\sqrt{\frac{2E_0}{m}} - \dot{x}}{\sqrt{\frac{2E_0}{m}} + \dot{x}}\right)\omega^1 + \mathrm{d}h$$

where $h(t,x,y)$ is a function on $Q_+$.

(c) $\bar{L} \neq 0$, $\bar{L}_2 \neq 0$, e.g.

$$\bar{\theta}_{(c)} = \frac{E_0}{m}\,\dot{x}^2\,\mathrm{d}t + \frac{2E_0}{m}\,\dot{x}\,\omega^1 - \left(mg^1 + (g^1)^3\right)\varphi^1 + \mathrm{d}h$$

where $h(t,x,y)$ is a function on $Q_+$.

**Comments.**

We have found three possible types of constraint Cartan forms that differ in functions $\bar{L}$ and $\bar{L}_2$ with concrete representatives $\bar{\theta}_{(a)}$, $\bar{\theta}_{(\mathrm{b})}$ and $\bar{\theta}_{(c)}$. The very interesting situation is that in which $\bar{L} = 0$. Such a situation represents an evidence of obvious fact that for non-holonomic systems a variational principle formulated by some authors (see e.g. [**1**]) in analogy to an unconstrained case, i.e. as the requirement of a stationary condition for the integral

$$\int_{t_1}^{t_2} J^1\gamma^*\bar{\theta} = \int_{t_1}^{t_2} J^1\gamma^*\bar{L}\mathrm{d}t$$

with $Q$-admissible sections $\gamma$ of $\pi$, does not work (cf. Remark 2). On the other hand, one can ask if the variational principle is allowed to be formulated by means of a special type of constraint Cartan form satisfying $\bar{L}_{m-k+i} = 0$, $1 \leqslant i \leqslant k$. However, such a form does not exist in general as it is shown in Example 2. Thus, a more general non-holonomic variational principle has to be used (see [**7**], [**18**]).

**Example 2.**

Consider a particle moving under Stokes force in the homogeneous gravitational field with the gravitational acceleration $\boldsymbol{G} = (G, 0)$. Unconstrained equations of motion

$$m\ddot{x} + \beta\dot{x} - mG = 0\,, \qquad m\ddot{y} + \beta\dot{y} = 0$$

are non-variational. The non-holonomic constraint representing the conservation of the mechanical energy can be written as follows:

$$\frac{1}{2}m\,(\dot{x}^2 + \dot{y}^2) - mGx = E_0 = const\,.$$

Reduced equations of motion on $Q_+ = \{a \mid a \in Q,\ \dot{y}(a) > 0\}$

$$g^1 = \dot{y} = \sqrt{\frac{2E_0}{m} + 2Gx - \dot{x}^2}\,, \qquad m\ddot{x}\left(1 + \frac{\dot{x}^2}{(g^1)^2}\right) - mG\left(1 + \frac{\dot{x}^2}{(g^1)^2}\right) = 0$$

are variational with a constraint Cartan form e.g.

$$\bar{\theta} = -2mGx\,\mathrm{d}t - \left(m\sqrt{\frac{2E_0}{m} + 2Gx - \dot{x}^2}\right)\varphi^1 + \mathrm{d}h$$

where $h(t,x,y)$ is a function on $Q_+$. Now, trying to find out a constraint Cartan form in which $\bar{L}_2 = 0$ we obtain system of equations for $\bar{L}$

$$\begin{aligned} m\left(1+\frac{\dot{x}^2}{(g^1)^2}\right) &= -\frac{\partial^2 \bar{L}}{\partial \dot{x}^2}\,, \\ -mG\left(1+\frac{\dot{x}^2}{(g^1)^2}\right) &= \frac{\partial_c \bar{L}}{\partial x} - \frac{\mathrm{d}'_c}{\mathrm{d}t}\frac{\partial \bar{L}}{\partial \dot{x}} \end{aligned}$$

that has no solution, i.e. there does not exist a constraint Cartan form such that $\bar{\theta} = \bar{L}\,\mathrm{d}t$. Similarly, one can check that also the function $\bar{L}$ has to be non-zero. Thus, $\bar{\theta}$ is a representative of the unique type of constraint Cartan forms.

## 4. Conclusion

In this paper we have discussed some variational properties of systems with non-holonomic constraints. More precise characterization of constraint Cartan forms and their classification is a subject of our next study.

**Acknowledgements.** Research is supported by grant 201/03/0512 of the Czech Grant Agency, by grant MSM 0021622409 of the Ministry of Education, Youth and Sports of the Czech Republic and by project 566 of the Fund of Development of Universities of the Ministry of Education, Youth and Sports of the Czech Republic.

## References

[1] Cardin, F., and Favretti, M., On nonholonomic and vakonomic dynamics of mechanical systems with nonintegrable constraints. *J. Geom. Phys.* **18** (1996), 295-325.

[2] Cariñena, J. F., and Rañada, M. F., Lagrangian systems with constraints: a geometric approach to the method of Lagrange multipliers. *J. Phys. A: Math. Gen.* **26** (1993), 1335-1351.

[3] Krupková, O., Mechanical systems with nonholonomic constraints. *J. Math. Phys.* **38** (1997), 5098-5126.

[4] Krupková, O., *The Geometry of Ordinary Variational Equations.* Lecture Notes in Mathematics **1678**, Springer, Berlin 1997.

[5] Krupková, O., On the geometry of non-holonomic mechanical systems. In *Differential Geometry and Applications.* Proc. conf., Brno, Czech Republic, 1998, edited by I. Kolář, O. Kowalski, D. Krupka, and J. Slovák (Masaryk University, Brno, 1999), 533-546.

[6] Krupková, O., Recent results in the geometry of constrained systems. *Reports on Math. Phys.* **49** (2002), 269-278.

[7] Krupková, O., and Swaczyna, M., *The non-holonomic variational principle.* Preprint, Masaryk University, Brno 2002, 34pp.

[8] Krupková, O., and Musilová, J., *Constraint Helmholtz conditions.* Preprint, Masaryk University, Brno 2002, 10pp.

[9] Krupková, O., and Musilová, J., Non-holonomic variational systems. Accepted for publication in *Reports on Math. Phys.*

[10] de León, M., Marrero, J. C., and de Diego, D. M., Non-holonomic Lagrangian systems in jet manifolds. *J. Phys. A: Math. Gen.* **30** (1997), 1167-1190.

[11] Massa, E., and Pagani, E., Classical mechanics of non-holonomic systems: a geometric approach. *Ann. Inst. Henri Poincaré* **55** (1991), 511-544.

[12] Massa, E., and Pagani, E., A new look at classical mechanics of constrained systems. *Ann. Inst. Henri Poincaré* **66** (1997), 1-36.

[13] Morando, P., and Vignolo, S., A geometric approach to constrained mechanical systems, symmetries and inverse problems. *J. Phys. A.: Math. Gen.* **31** (1998), 8233-8245.

[14] Rañada, M. F., Time-dependent Lagrangian systems: A geometric approach to the theory of systems with constraints. *J. Math. Phys.* **35** (1994), 748-758.

[15] Sarlet, W., A direct geometrical construction of the dynamics of non-holonomic Lagrangian systems. *Extracta Mathematicae* **11** (1996), 202-212.
[16] Sarlet, W., Cantrijn, F., and Saunders, D. J., A geometrical framework for the study of non-holonomic Lagrangian systems. *J. Phys. A.: Math. Gen.* **28** (1995), 3253-3268.
[17] Saunders, D. J., Sarlet, W., and Cantrijn, F., A geometrical framework for the study of non-holonomic Lagrangian systems II. *J. Phys. A.: Math. Gen.* **29** (1996), 4265-4274.
[18] Swaczyna, M., *On the non-holonomic variational principle.* International Workshop on Global Analysis, Cankaya University Ankara, Turkey, April 15-17, 2004 (accepted for publication).

Faculty of Science, Masaryk University,
Kotlářská 2, 611 37 Brno,
Czech Republic