DIFFERENTIAL GEOMETRY AND ITS APPLICATIONS
Proc. Conf. Prague, August 30 – September 3, 2004
Charles University, Prague (Czech Republic), 2005, 437 – 452

Lie systems and Connections in fibre bundles: Applications in Quantum Mechanics

José F. Cariñena and Arturo Ramos

Abstract. We describe the geometry of Lie systems, that is systems of differential equations admitting a superposition function for the general solution in terms of a set of particular solutions and some constants, in terms of connections on principal and associated bundles. The theory allows us to understand the reduction theorem for Lie systems on groups in terms of the group of automorphisms of the relevant principal bundle. The theory is illustrated with some applications, especially in time-dependent quantum mechanical problems.

1. Introduction

The use of modern geometric tools has been very useful in the study of differential equations and dynamical systems and also in many fields of classical and quantum theories of physics. In particular the theory of the so-called Lie systems has received a lot of attention during the last years maybe because of its many applications not only in the above mentioned branches of physics but also in many other different fields as, for instance, in control theory. We recall that for Lie systems we understand nonautonomous systems of differential equations

$$\frac{dx^i}{dt} = X^i(x^1, \dots, x^n, t), \quad i = 1 \dots, n, \tag{1}$$

for which the general solution can be expressed as a function $\Phi : \mathbb{R}^{n(m+1)} \to \mathbb{R}^n$ of a fundamental set of m solutions and the n constants characterizing each solution:

$$x = \Phi(x_{(1)}, \dots, x_{(m)}; k_1, \dots, k_n)\,.$$

These systems arise very frequently in classical physics and have interesting analogue in quantum physics and therefore they deserves special attention. Actually the main advantage is that we can deal with both the classical and the quantum problem in parallel without any conceptual difference. The characterization of such systems was given by Lie [**1**].

From the geometric viewpoint the system of first-order differential equations (1) provides the integral curves of the t-dependent vector field on an n-dimensional manifold M

$$X(x,t) = \sum_{i=1}^{n} X^i(x,t)\,\frac{\partial}{\partial x^i}$$

This paper is in final form and no version of it will be published elswhere.

in the same way as it happens for autonomous systems and true vector fields (see, for example, [**2**]). Lie characterized the systems admitting such nonlinear superposition rule as those which can be written as a t-dependent linear combination of vector fields,

$$X(x,t)=\sum_{\alpha=1}^{r} Z_\alpha(t)\,Y^{(\alpha)}(x)\ , \tag{2}$$

with the vector fields $Y^{(\alpha)}$ closing on a finite-dimensional real Lie algebra. They are called Lie systems and have been the subject of a number of works by Anderson, Harnad, Winternitz and collaborators, who also studied their applications in Physics and Mathematics [**3, 4, 5, 6**]. Both homogeneous and nonhomogeneous linear systems are particular instances of Lie systems, but also another very interesting example, for $n=1$, is the Riccati equation

$$\frac{dx}{dt}=b_2(t)\,x^2(t)+b_1(t)\,x(t)+b_0(t)\ , \tag{3}$$

for which $m=3$ and the superposition function comes from the relation

$$\frac{x-x_{(1)}}{x-x_{(2)}}:\frac{x_{(3)}-x_{(1)}}{x_{(3)}-x_{(2)}}=k$$

or, in other words,

$$x=\frac{x_{(1)}(x_{(3)}-x_{(2)})+k\,x_{(2)}(x_{(1)}-x_{(3)})}{(x_{(3)}-x_{(2)})+k\,(x_{(1)}-x_{(3)})}\ .$$

More information on general Lie systems and applications can be found in [**7, 8, 9**].

Our aim is to show that all these Lie systems can be considered as the systems defining the horizontal curves with respect to some connections in bundles and then we use some properties of the theory of such connections to reduce the systems to simpler ones. This reduction is used not only in classical physics but also in the study of the dynamical evolution defined by some t-dependent quantum Hamiltonians.

The paper is organized as follows: The notation and basic definitions are introduced in Section 2, in which the fundamental properties of Lie systems are also described. The main mathematical result is established in Section 3 where we show that these Lie systems arise in a natural way as describing horizontal curves with respect to a connection. The characterization of the group of automorphisms of the bundle is given in Section 4 and the rôle of this group in the reduction is revisited from this new perspective. Section 5 is devoted to comment some of the recent applications of the theory in Supersymmetric Quantum Mechanics. Lie systems of Hamiltonian type, both in Classical and Quantum Mechanics are studied in Section 6. Finally the applications of the theory in several examples illustrating the use of these Lie systems in quantum mechanics, in particular the time evolution of some quantum systems described by time-dependent Hamiltonians, are given in Section 7.

2. Notation and basic definitions

Let G be a r-dimensional connected Lie group and $e\in G$ its neutral element. A basis of T_eG is denoted $\{a_1,\dots,a_r\}$. The Lie group G acts on itself in a transitive and free way both on the left and on the right

$$L_g : G \to G\ , \qquad R_g : G \to G$$
$$g' \mapsto g\,g'\ , \qquad g' \mapsto g'\,g\ .$$

The diffeomorphisms L_g and R_g allow us to define the left- and right-invariant vector fields determined by their values in the neutral element $X^L_g = L_{g*e}X_e$ and $X^R_g = R_{g*e}X_e$. We use the shorter notation $X^L_\alpha = X^L_{a_\alpha}$ and $X^R_\alpha = X^R_{a_\alpha}$, $\alpha = 1, \dots, r$, i.e., $(X^L_\alpha)_g = L_{g*e}(a_\alpha)$, $(X^R_\alpha)_g = R_{g*e}(a_\alpha)$. They generate, respectively, two Lie algebras

$$[X^L_\alpha, X^L_\beta] = \sum_{\gamma=1}^r c_{\alpha\beta}{}^\gamma\, X^L_\gamma\ , \quad [X^R_\alpha, X^R_\beta] = -\sum_{\gamma=1}^r c_{\alpha\beta}{}^\gamma\, X^R_\gamma\ .$$

The inversion map $\psi : G \to G$ given by $\psi(g) = g^{-1}$ is such that ψ_* is a Lie algebra isomorphism of these two opposite Lie algebras.

Given a curve in G, $\gamma : \mathbb{R} \to G$, $t \mapsto g(t)$, we denote by $\dot g(t)$ the vector field along the curve $g(t)$ defined by the tangent vector at each point of the curve. Such a curve gives rise to two families of curves filling all G, $\{L_{g_0} \circ \gamma \mid g_0 \in G\}$ and $\{R_{g_0} \circ \gamma \mid g_0 \in G\}$. In particular, if $g(t)$ starts from $g(0) = e$, then, for each $g_0 \in G$, $\bar g(t) = g(t)\, g_0$ is a curve starting from g_0 (the right-translation by g_0, $R_{g_0} \circ \gamma$, of the curve γ given by $g(t)$) and similarly $\bar{\bar g}(t) = g_0\, g(t)$ is a curve starting from g_0 (the left-translation by g_0, $L_{g_0} \circ \gamma$, of the curve γ). Moreover, as $\dot g(t)$ is a vector field along the curve $g(t)$, for each value of the parameter t,

$$\dot g(t)\, g^{-1}(t) = -\sum_{\alpha=1}^r b_\alpha(t) a_\alpha \in T_eG\ . \tag{4}$$

The left hand side is to be understood as $R_{g^{-1}(t)*g(t)}(\dot g(t))$. When $G \subset GL(n, \mathbb{R})$, it reduces to the given expression.

The curve $g(t)$ can be seen to be determined by the curve in T_eG

$$a(t) = -\sum_{\alpha=1}^r b_\alpha(t) a_\alpha$$

by means of the preceding relation and the initial condition $g(0) = e$. More explicitly $g(t)$ is the integral curve starting from $e \in G$ of the t-dependent vector field in G

$$\overline{X}(g, t) = -\sum_{\alpha=1}^r b_\alpha(t) X^R_\alpha(g)\ , \tag{5}$$

which defines a Lie system in the Lie group G, for every choice of the functions $b_\alpha(t)$. Actually this is the most general form of a right-invariant Lie system in the Lie group G.

In fact the integral curves $g(t)$ of such t-dependent vector field are the solutions of the system

$$\dot g = -\sum_{\alpha=1}^r b_\alpha(t)\, X^R_\alpha(g)\ .$$

When applying $R_{g^{-1}(t)*g(t)}$ to both sides we obtain equation (4).

Note that this equation is right-invariant: if $g(t)$ is the solution such that $g(0) = e$, then, for each $g_0 \in G$, $\bar g(t) = g(t)\, g_0$ is the solution such that $\bar g(0) = g_0$ and consequently the solution curves of the system are obtained from just one solution by right-translations. In fact

$$\begin{aligned}\dot{\bar{g}}(t) &= R_{g_0*g(t)}(\dot{g}(t)) = R_{g_0*g(t)}\left[R_{g(t)*e}\left(-\sum_{\alpha=1}^{r} b_\alpha(t)\,a_\alpha\right)\right]\\ &= R_{(g(t)g_0)*e}\left(-\sum_{\alpha=1}^{r} b_\alpha(t)\,a_\alpha\right) = -\sum_{\alpha=1}^{r} b_\alpha(t)X_\alpha^R(\bar{g}(t))\ .\end{aligned}$$

Finally we also recall that the set $\mathcal{G}$ of the curves $\gamma : \mathbb{R} \to G$, $t \mapsto g(t)$, is a group with respect to the point-wise composition law:

$$\gamma_2 * \gamma_1 : t \mapsto g_2(t)\, g_1(t)\ .$$

Let $\Phi : G \times M \to M$ define a left-action of the Lie group G on a differentiable manifold M. We denote: $gu := \Phi_g(u) := \Phi(g,u)$; $\Phi_u(g) := \Phi(g,u)$. Note that the maps Φ_g are diffeomorphisms and that $(\Phi_g)^{-1} = \Phi_{g^{-1}}$. Denote by $\mathfrak{g}$ the Lie algebra of G, i.e. the set of left-invariant vector fields in G. Observe that $\Phi_{u*e} : \mathfrak{g} \cong T_eG \to T_uM$. The map $X : \mathfrak{g} \to \mathcal{X}(M)$ defined by $a \mapsto X_a(u) = \Phi_{u*e}(-a_e)$ defines a mapping of $\mathfrak{g}$ into $\mathcal{X}(M)$. We call X_a the fundamental vector field associated to the element a of $\mathfrak{g}$ which is given by

$$(X_af)(u) = \frac{d}{dt} f(\exp(-ta)u)\Big|_{t=0}\ , \quad f \in C^\infty(M)\ .$$

Moreover the minus sign has been introduced for X to be a Lie algebra homomorphism, i.e. $X_{[a,b]} = [X_a, X_b]$. The flow of the vector field $X_a \in \mathcal{X}(M)$ is $\varphi(t,u) = \Phi(\exp(-ta), u)$.

A particularly important case is when we consider a transitive action. Let H be a closed subgroup of G and consider the homogeneous space $M = G/H$, which admits one differentiable structure for which the canonical projection $\tau : G \to G/H$ is a differentiable map and admits local differentiable sections. Then G acts on G/H by $\lambda(g_2, g_1H) = (g_2g_1)H$ and moreover G can be seen as a principal bundle over G/H: $(G, \tau, G/H)$.

The right-invariant vector fields X_α^R are τ-projectable [**7**]: the τ-related vector fields in M are the fundamental vector fields $-X_\alpha = -X_{a_\alpha}$ corresponding to the natural left-action of G on M:

$$\tau_{*g} X_\alpha^R(g) = -X_\alpha(gH)\ .$$

The t-dependent vector field in M projection of (5) is

$$X(x,t) = \sum_{\alpha=1}^{r} b_\alpha(t) X_\alpha(x)\ , \tag{6}$$

where the vector fields X_α close on a Lie algebra isomorphic to $\mathfrak{g}$, and its integral curves are the solutions of the system of differential equations:

$$\dot{x} = \sum_{\alpha=1}^{r} b_\alpha(t) X_\alpha(x)\ .$$

Therefore a solution of this last system starting from x_0 is:

$$x(t) = \Phi(g(t), x_0)$$

with $g(t)$ being the solution of (4) starting form e. Note that, as the vector fields X_α close on a Lie algebra isomorphic to $\mathfrak{g}$, the systems of differential equations we obtain are those characterized by Lie as admitting a superposition function for the general solution in terms of a fundamental set of solutions [**1, 7, 8, 9**].

3. Lie systems and connections on bundles

If G is a r-dimensional connected Lie group, then $\pi_2 : P = G\times\mathbb{R}\to\mathbb{R}$ is a trivial principal G-bundle over the base $\mathbb{R}$. The right-action of G on P is given by

$$\Phi((g',t),g)=\Phi_g((g',t))=(g'\,g,t)\ .$$

Giving a connection in the principal bundle P is equivalent to giving a curve in G, for instance, one such that $g(0)=e$. This curve provides us firstly a section for π_2, $\sigma(t)=(g(t),t)$, and then a family of sections which are right-translated from such a section,

$$\{\sigma_{g_0}(t)=\sigma(t)\,g_0\mid g_0\in G\}\ .$$

The tangent vectors to such a family of sections span the horizontal spaces in each point. More specifically vertical and horizontal spaces in a point of P are given, respectively, by:

$$VP_{(g_0,t)}=\ker\pi_{2*(g_0,t)}=\langle(X^R_\alpha(g_0),0)\rangle\ ,\quad HP_{(g_0,t)}=\langle(R_{g_0*e}(\dot g(t)\,g^{-1}(t)),1)\rangle\ .$$

Note that this choice of horizontal subspaces is right-invariant because

$$\begin{aligned}R_{g_1*(g_0,t)}HP_{(g_0,t)}&=\langle(R_{g_1*g_0}R_{g_0*e}(\dot g(t)\,g^{-1}(t)),1)\rangle\\&=\langle(R_{g_0g_1*e}(\dot g(t)\,g^{-1}(t)),1)\rangle=HP_{(g_0g_1,t)}\end{aligned}$$

and also that

$$\Phi_{g^{-1}(t)*g(t)}(\dot g(t),1)=(R_{g^{-1}(t)*g(t)}\dot g(t),1)=(\dot g(t)\,g^{-1}(t),1)\ .$$

The choice of the connection given by γ amounts to choosing a basis of the tangent space at the point (g_0,t) as follows:

$$\{X^R_\alpha(g_0),\,\partial/\partial t+R_{g_0*e}(\dot g(t)\,g^{-1}(t))\mid\alpha=1,\dots,r\}\ ,$$

the dual basis being

$$\{\theta^R_\alpha(g_0)+\tau_\alpha(t)\,dt,\,dt\mid\alpha=1,\dots,r\}\,,$$

where the coefficients $\tau_\alpha(t)$ are determined by

$$R_{g_0*e}(\dot g(t)\,g^{-1}(t))=-\sum_{\beta=1}^r\tau_\beta(t)\,X^R_\beta(g_0)\ ,$$

i.e.

$$\tau_\alpha(t)=\langle\theta^R_\alpha(g_0),-R_{g_0*e}(\dot g(t)\,g^{-1}(t))\rangle\ ,$$

and the associated vertical projector is

$$P^v_{(g_0,t)}=\sum_{\beta=1}^r X^R_\beta(g_0)\otimes(\theta^R_\beta(g_0)+\tau_\beta(t)\,dt)=I_{T_{g_0}G}-\big(R_{g_0*e}(\dot g(t)\,g^{-1}(t))\big)\,dt\ .$$

Conversely any principal connection in P is given by an invariant distribution. It is determined by the horizontal spaces

$$HP_{(e,t)}=\langle(a(t),1)\rangle\ ,$$

where, for each t, $a(t) \in T_eG$ is given

$$a(t) = -\sum_{\alpha=1}^{r} b_\alpha(t) a_\alpha$$

with $b_\alpha(t)$ being arbitrary functions of t, and then

$$HP_{(g_0,t)} = \langle (R_{g_0*e}(a(t)), 1) \rangle \ .$$

Therefore we see that there is a one-to-one correspondence between principal connections in the principal bundle $\pi_2 : P = G \times \mathbb{R} \to \mathbb{R}$ and Lie systems in the Lie group G: the curve $t \mapsto g(t)$ is a solution of the Lie system if and only if the section $\sigma : t \mapsto (g(t), t)$ is an integral 1-dimensional surface of the corresponding distribution in P.

As is shown later, we can make use the group of automorphisms of the bundle for relating different connections and therefore for relating different Lie systems. The main point is to find, for a given Lie system, an appropriate automorphism transforming the Lie system into another, simpler, one. This leads to the theory of reduction for Lie systems. Our aim is to show in a later section that the theory of Lie systems in homogeneous spaces for the Lie group G plays an important rôle in such a process of reduction.

A left action $\Psi : G \times M \to M$ defines an associated bundle E with base $\mathbb{R}$ and typical fibre M. The total space of the bundle, E, is the set of orbits of the right-action of G on $P \times M$,

$$(u, x)g = (\Phi(u, g), \Psi(g^{-1}, x)) \ ,$$

and the projection is $\pi_E[u, x] = \pi_2(u)$, where $[u, x]$ denotes the equivalence class of $(u, x) \in P \times M$. Note that $[\Phi(u, g), x] = [u, \Psi(g, x)]$.

A connection in the principal bundle translates into a connection in the associated bundle E and the horizontal curves are then $[\widetilde{\gamma}(t), x]$, where $\widetilde{\gamma}(t)$ is an horizontal curve in P. More explicitly, as the curves $\widetilde{\gamma}$ are of the form $\widetilde{\gamma}(t) = (g(t)g_0, t)$, we find that the horizontal curves in the associated bundle are

$$[(g(t)g_0, t), x_0] = [\Phi((e, t), g(t)g_0), x_0] = [(e, t), \Psi(g(t)g_0, x_0)]$$

and consequently

$$[(g(t)g_0, t), x_0] = [(e, t), \Psi(g(t), \Psi(g_0, x_0))] \ .$$

Remark that E is equivalent to the product $E = M \times \mathbb{R}$: $[(e, t), x]$ corresponds to (x, t). With this identification the horizontal curves here considered correspond to the integral curves starting from the points $\Psi(g_0, x_0)$ of the associated Lie system in M with respect to the action of G on M given by Ψ. The simplest case is when $M = V$ is a linear space and a linear representation of G on the vector space V is considered, the associated bundle being then a vector bundle and the corresponding Lie system being a linear system. This means that a linear system of differential equations can be seen as defining the horizontal curves corresponding to a connection in an associated vector bundle, as pointed out in [**10**] (see also [**11**] for the case of Schrödinger equation). Note that different linear representations give rise to different associated vector bundles and, correspondingly, to different systems of differential equations.

The existence of many different Lie systems in homogeneous spaces for the Lie group G that are associated with the same Lie system in the Lie group G appears here as corresponding to the fact that that the principal bundle $\pi_2 : P = G\times\mathbb{R} \to \mathbb{R}$ can have many different associated bundles.

4. The group of automorphisms of a principal bundle and the reduction method

Let now $\Psi : P \to P$ be an automorphism of a principal G-bundle, i.e. a fibre-preserving diffeomorphism over the identity map commuting with the right G-action. So, if the right-action of G on P is given by $\Phi : P\times G \to G$, then $\Psi\circ\Phi_g = \Phi_g\circ\Psi$. If H is a right-invariant distribution on P, then we can define a new distribution on P by $K_{\Psi(u)} = \Psi_{*u}(H_u)$. Such distribution is right-invariant because, for any $g \in G$,

$$K_{\Phi(\Psi(u),g)} = K_{(\Phi_g\circ\Psi)(u)} = K_{(\Psi\circ\Phi_g)(u)} = K_{\Psi(\Phi(u,g))} = \Psi_{*\Phi(u,g)}(H_{\Phi(u,g)})$$

and taking into account the right-invariance of the distribution H, $H_{\Phi(u,g)} = \Phi_{g*u}(H_u)$ and the commutativity of Φ_g with Ψ, we obtain

$$K_{\Phi(\Psi(u),g)} = (\Psi_{*\Phi(u,g)}\circ\Phi_{g*u})(H_u) = (\Psi\circ\Phi_g)_{*u}(H_u) = (\Phi_g\circ\Psi)_{*u}(H_u)$$

and then

$$K_{\Phi(\Psi(u),g)} = \Phi_{g*\Psi(u)}(\Psi_{*u}(H_u)) = \Phi_{g*\Psi(u)}(K_{\Psi(u)})\ ,$$

which shows that the distribution defined by K is right-invariant. On the other hand, as Ψ is a fibre-preserving map over the identity, the image under Ψ of vertical vectors are vertical vectors and therefore, if H is a horizontal distribution defining a connection in P, the new distribution K is also horizontal and defines a new connection on P. In this way we define an action of the group of automorphisms of the principal G-bundle on the set of its principal connections.

In the particular case of $P = G\times\mathbb{R}$ considered before, the G-bundle automorphisms are given by maps $\Psi(g,t) = (\psi(g,t),t)$ and the condition, $\Psi\circ\Phi_{g'} = \Phi_{g'}\circ\Psi$ for any $g'\in G$, means that ψ is such that $\psi(g,t)g' = \psi(gg',t)$. In particular taking $g = e$ and $g' = g_0$ we obtain that $\psi(g_0,t) = \psi(e,t)g_0$. In other words every automorphism Ψ is determined by a curve $\gamma : \mathbb{R}\to G$, $g(t) = \psi(e,t)$ by means of $\psi(g_0,t) = (g(t)\,g_0,t)$. Conversely any curve γ in G given by $t\mapsto g(t)$ defines an automorphism of the G-bundle $P = G\times\mathbb{R}$: $\Psi(g_0,t) = (g(t)g_0,t)$. In this way we can identify the group of automorphisms of the G-bundle with the above mentioned group $\mathcal{G}$ of curves in G and with this identification the action of the automorphism is just left-multiplication by $g(t)$ of the G-component.

We can make use of this action of the group $\mathcal{G}$ on the set of Lie systems in order to relate a given problem with other similar but maybe simpler Lie systems obtained by such automorphisms as follows. We firstly remark that, given an equation on a Lie group like (4), it may happen that the only nonvanishing coefficients are those of a subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ and the equation reduces to a simpler one on a subgroup, involving fewer coordinates.

Consider an equation like (4) and choose a curve $g'(t)$ in the group corresponding to a given automorphism, $\Psi(g,t) = (g'(t)\,g,t)$, and define the curve $\overline{g}(t)$ by $\overline{g}(t) = g'(t)g(t)$. The new curve in G, $\overline{g}(t)$, determines a new connection and therefore a new Lie system. Indeed

$$R_{\overline{g}(t)^{-1}*\overline{g}(t)}(\dot{\overline{g}}(t)) = R_{g'^{-1}(t)*g'(t)}(\dot g'(t)) - \sum_{\alpha=1}^{r} b_\alpha(t)\mathrm{Ad}(g'(t))a_\alpha\ , \tag{7}$$

which is an equation similar to the original one but with a different right hand side.

Therefore the aim is to choose the curve $g'(t)$ in such a way that the new equation be simpler in the preceding sense. Once a subgroup H has been chosen we should look for an appropriate choice of $g'(t)$ such that the right hand side lies in T_eH and hence $\overline{g}(t) \in H$ for all t. The fundamental result, the proof of what can be found in [**7**], is that the knowledge of a particular solution of the associated Lie system in a homogeneous space G/H allows us to reduce the problem to an analogous one but in the subgroup H.

If $\Phi : G \times M \to M$ is a transitive action of G on a homogeneous space M, which can be identified with the set G/H of left-cosets by choosing a point at M, then the horizontal curve $x(t)$ starting from the point $x_0 = x(0)$ and the horizontal curve $\bar{x}(t)$ starting from $\Phi(g'(0), x_0)$, associated with the connections defined by $g(t)$ and $\bar{g}(t)$, respectively, are related by

$$\overline{x}(t) = \Phi(\overline{g}(t), x_0) = \Phi(g'(t)g(t), x_0) = \Phi(g'(t), x(t)) \ .$$

Therefore the action of the group of curves in G on the set of connections translates to the homogeneous space and gives an action on the corresponding set of associated Lie systems.

More explicitly, if we consider the automorphism Ψ of P defined by $g'(t)$, the Lie system (6) transforms into a new one

$$\dot{\bar{x}} = \sum_{\alpha=1}^{r} \bar{b}_\alpha(t) X_\alpha(\bar{x}) \ ,$$

in which $\sum_{\alpha=1}^{r} \bar{b}_\alpha a_\alpha$ is the element of the Lie algebra such that

$$\bar{b} = \mathrm{Ad}(g'(t))b(t) - \dot{g}'\, g'^{-1} \ .$$

THEOREM 1. *Each solution of (4) on the group G can be written in the form $g(t) = g_1(t)\, h(t)$, where $g_1(t)$ is a curve on G projecting onto a solution $\tilde{g}_1(t)$ for the left action λ on the homogeneous space G/H and $h(t)$ is a solution of an equation but for the subgroup H, given explicitly by*

$$(\dot{h}\, h^{-1})(t) = -\mathrm{Ad}(g_1^{-1}(t)) \left(\sum_{\alpha=1}^{r} b_\alpha(t) a_\alpha + (\dot{g}_1\, g_1^{-1})(t) \right) \in T_eH \ .$$

By iteration of this procedure the knowledge of more particular solutions of the same Lie system in a given homogeneous space, characterized by different initial conditions, allows us to reduce the original Lie system in G to one on the Lie subgroup of G obtained by intersection of the corresponding isotopy subgroups of such points. In this way the knowledge of a large enough number of solutions reduces the problem to one on the improper group of one element and therefore there is no need to do any integration. We obtain in this way a superposition function for the solutions of the given differential equation of Lie type.

We now consider the case in which the Lie group G is a semidirect product $G = G_1 \odot G_2$, i.e., G_1 is a normal subgroup in G, G_2 is a subgroup of G, and each element of G can be factorized in a unique way as a product $g = g_2\, g_1$, where $g_1 \in G_1$ and $g_2 \in G_2$. In such a case, if we consider the curve in G given by $g(t) = g_2(t) g_1(t)$, we see that

$$R_{g(t)^{-1}*g(t)}(\dot{g}(t)) = R_{[g_2(t)g_1(t)]^{-1}*g_2(t)g_1(t)}(\dot{g}_2(t)g_1(t) + g_2(t)\dot{g}_1(t)) \ .$$

The first term on the right hand side is

$$R_{[g_2(t)g_1(t)]^{-1}*g_2(t)g_1(t)}R_{g_1(t)*g_2(t)}\dot{g}_2(t) = R_{g_2(t)^{-1}*g_2(t)}\dot{g}_2(t)\ ,$$

where use has been made of the relation

$$R_{gh*k} = (R_h \circ R_g)_{*k} = R_{h*kg} \circ R_{g*k}\ , \qquad g, h, k \in G\,,$$

for $g = g_1(t)$, $h = (g_2(t)g_1(t))^{-1}$ and $k = g_2(t)$. The second term in the right hand side can be written as

$$R_{[g_2(t)g_1(t)]^{-1}*g_2(t)g_1(t)}\left(L_{g_2(t)*g_1(t)}\dot{g}_1(t)\right) = L_{g_2(t)*g_1(t)[g_2(t)g_1(t)]^{-1}}R_{[g_2(t)g_1(t)]^{-1}*g_1(t)}(\dot{g}_1(t))$$

and therefore as

$$L_{g_2(t)*g_2(t)^{-1}}R_{g_1(t)^{-1}g_2(t)^{-1}*g_1(t)}(\dot{g}_1(t)) = L_{g_2(t)*g_2(t)^{-1}}R_{g_2(t)^{-1}*e}R_{g_1(t)^{-1}*g_1(t)}(\dot{g}_1(t))\ ,$$

which using the definition of $\mathrm{Ad}\, g$ reduces to $\mathrm{Ad}(g_2(t))_e(R_{g_1(t)^{-1}*g_1(t)}(\dot{g}_1(t))$. This means that the solution of the original equation for $g(t)$, $R_{g(t)^{-1}*g(t)} = a(t)$, reduces to firstly solving an analogous equation for the component $g_2(t)$, but where the right-hand side is the component $a_2(t)$ of the decomposition $a(t) = a_1(t) + a_2(t)$ with $a_i(t) \in \mathfrak{g}_i$, and then solving another similar equation for $g_1(t)$ with the second member replaced by $\mathrm{Ad}\,(g_2(t)^{-1})a_1(t)$. This curve takes values in $\mathfrak{g}_1$ because we have assumed that G_1 is a normal subgroup in G and therefore $\mathfrak{g}_1$ is an ideal of $\mathfrak{g}$. Consequently, when G is a semidirect product $G = G_1 \odot G_2$, the solution of the Lie systems in G reduces to the solution of two Lie systems in each subgroup. The simplest case is when G is a direct product because then both subsystems are unrelated.

Finally the theory of reduction shows that even in the more general case in which G is not a semidirect product, but there exists a normal Lie subgroup G_1, the problem reduces to solving a particular solution of a Lie system in $G_2 = G/G_1$ and the general solution in G_1. Of course, when G_1 is a Lie subgroup but it is not an invariant subgroup, the solution of Lie systems in G amounts to finding a particular solution of a Lie system in the homogeneous space G/G_1 and the solution of a Lie system in the subgroup G_1.

Before ending this section we mention that there exists a method of dealing with equation (4) (or the corresponding equation in the subgroup once the reduction procedure has been carried out). This method is a generalization of the procedure proposed by Wei and Norman [**12, 13**] for finding the time evolution operator for a linear system of type $dU(t)/dt = H(t)U(t)$, with $U(0) = I$, see also [**14**]. More details can be found, for instance, in [**7, 8, 15**].

5. Applications of Lie systems in Supersymmetric Quantum Mechanics

As was indicated in the Introduction, the simplest example, and in some sense a prototype, of Lie systems is that of a linear system, but the Riccati equation is also an interesting example for $n = 1$ [**16**]. Both nonautonomous linear systems and Riccati equations appear very often in Physics. For instance linear systems appear in the time evolution of classical time-dependent harmonic oscillators and related problems. Also in Quantum Mechanics one often considers finite-dimensional Hilbert spaces, for instance when only the internal degrees of freedom are taken into account [**17**] and then the time-dependent Schrödinger equation reduces to a linear system. On the other hand the Riccati equation appears very often in problems of Supersymmetric Quantum Mechanics, being the equation determining the superpotential function W

in the factorization of a Hamiltonian H and it also plays a very relevant rôle in the search for shape-invariant potentials [**18**] (see also [**19**] for a modern approach).

Another remarkable fact is that Lie reduction theory for the invariance under dilations of linear second-order differential equations produces a Riccati equation [**14**]. In the particular case of the time-independent Schrödinger equation,

$$-\frac{d^2\psi}{dx^2} + (V(x) - E)\psi = 0 \ ,$$

the reduction leads to the first order equation for $\varphi = \psi'/\psi$:

$$\frac{d\varphi}{dx} = -\varphi^2 + (V - E) \ .$$

This suggests that we can make use of the theory of Riccati equations to study spectral problems of some quantum Hamiltonians.

The general theory of Lie systems in the particular example of the Riccati equation tells us that each Riccati equation $\dot{x} = b_2(t)\, x^2 + b_1(t)\, x + b_0(t)$ can be considered as a curve in $\mathbb{R}^3$, identified to the Lie algebra of $\mathfrak{sl}(2,\mathbb{R})$ by the choice of a basis, $b(t) = (b_2(t), b_1(t), b_0(t))$, and then defining a connection as indicated before. Any automorphism of the corresponding principal bundle $\pi : SL(2,\mathbb{R}) \times \mathbb{R} \to \mathbb{R}$, i.e. a curve in $SL(2,\mathbb{R})$ given by

$$A = \begin{pmatrix} \alpha(t) & \beta(t) \\ \gamma(t) & \delta(t) \end{pmatrix} \in \mathrm{Map}\,(\mathbb{R},\ SL(2,\mathbb{R})) \ ,$$

transforms such a connection giving rise to a new Riccati equation in which the coefficients are related to the original one as follows [**16**]:

$$\begin{aligned} \bar{b}_2 &= \delta^2\, b_2 - \delta\,\gamma\, b_1 + \gamma^2\, b_0 + \gamma\,\dot{\delta} - \delta\,\dot{\gamma} \\ \bar{b}_1 &= -2\beta\,\delta\, b_2 + (\alpha\,\delta + \beta\,\gamma)\, b_1 - 2\,\alpha\,\gamma\, b_0 + \delta\,\dot{\alpha} - \alpha\,\dot{\delta} + \beta\,\dot{\gamma} - \gamma\,\dot{\beta} \\ \bar{b}_0 &= \beta^2\, b_2 - \alpha\,\beta\, b_1 + \alpha^2\, b_0 + \alpha\,\dot{\beta} - \beta\,\dot{\alpha}\, . \end{aligned}$$

This defines an affine action of the group $\mathcal{G}$ of curves in $SL(2,\mathbb{R})$ on the set of Riccati equations and we can use this action either for obtaining solutions of a Riccati equation by transforming the original equation into a simpler one in the same orbit or also for establishing some solubility criteria [**16**]. This action can also be used for relating spectral problems of two Hamiltonians the associated Riccati equations of which are connected by such $\mathcal{G}$-action as was done in [**20**].

6. Hamiltonian Lie systems in Classical and Quantum Mechanics

In the geometric approach to Classical Mechanics one considers a symplectic manifold (M, Ω), for instance the phase space $M = T^*Q$ of a configuration space Q endowed with its canonical exact symplectic structure. Infinitesimal canonical transformations are given by (locally) Hamiltonian vector fields, X, with flows preserving the symplectic form Ω. In particular time evolution is given by a Hamiltonian vector field. Therefore we should consider the case of Lie systems in which the vector fields arising in the expression of the t-dependent vector field are Hamiltonian vector fields closing on a real finite-dimensional Lie algebra. These vector fields (if satisfying the technical condition of being complete vector fields) are the fundamental vector fields relative to a symplectic action of the group G on the symplectic manifold (M, Ω). Note that the Hamiltonian functions of such vector fields, defined by

$i(X_\alpha)\Omega = -dh_\alpha$, however, do not close on the same Lie algebra as the vector fields X_α when the Poisson Bracket is considered, but we can only say that

$$d\left(\{h_\alpha, h_\beta\} - h_{[X_\alpha, X_\beta]}\right) = d\left(\{h_\alpha, h_\beta\} - c_{\alpha\beta}{}^{\gamma}\, h_\gamma\right) = 0$$

and therefore they span a Lie algebra which is a central extension of the original one.

For instance an example illustrating the possible applications of the theory is the motion of a classical particle under the action of a linear potential. The classical Hamiltonian is

$$H = \frac{p^2}{2m} + f(t)\, x\ .$$

For instance, when $f(t) = q\,E_0 + q\,E\,\cos\omega t$, the Hamiltonian describes the motion of a particle of electric charge q and mass m driven by a monochromatic electric field. The Hamilton equations of motion are

$$\begin{aligned} \dot{x} &= \frac{p}{m}\,, \\ \dot{p} &= -f(t)\,. \end{aligned} \tag{8}$$

The geometric approach using the theory of Lie systems was recently developed in [**15**]. The t-dependent vector field describing the time evolution

$$X = \frac{p}{m}\frac{\partial}{\partial x} - f(t)\,\frac{\partial}{\partial p}\ ,$$

can be written as a linear combination $X = (1/m)\,X_1 - f(t)\,X_2$, with

$$X_1 = p\,\frac{\partial}{\partial x}\ , \qquad X_2 = \frac{\partial}{\partial p}$$

being two vector fields closing with $X_3 = \partial/\partial x$ a three-dimensional Lie algebra isomorphic to the Heisenberg algebra, namely,

$$[X_1, X_2] = -X_3\ , \qquad [X_1, X_3] = 0\ , \qquad [X_2, X_3] = 0\ . \tag{9}$$

The three vector fields X_1, X_2 and X_3 are Hamiltonian vector fields with respect to the usual symplectic structure, $\Omega = dx \wedge dp$, the corresponding Hamiltonian functions h_α such that $i(X_\alpha)\Omega = -dh_\alpha$ being

$$h_1 = -\frac{p^2}{2}\,, \qquad h_2 = x\,, \qquad h_3 = -p\ .$$

Therefore

$$\{h_1, h_2\} = -h_3\,, \quad \{h_1, h_3\} = 0\,, \quad \{h_2, h_3\} = -1\,, \tag{10}$$

which close on a four-dimensional Lie algebra with $h_4 = 1$, that is, a central extension of that given by (9).

This is an important point: as indicated above and we see next, when analyzing the analogous problems in Quantum Mechanics, the group playing a rôle is not the same as in the classical case but this kind of extension. All the other aspects are similar and the propagator in the corresponding quantum problem may be determined by the Wei–Norman method with the replacement of the old group by the new one.

As far as Quantum Mechanics is concerned, we firstly remark that the separable complex Hilbert space of states $\mathcal{H}$ can be seen as a (infinite-dimensional) real manifold admitting a global chart [**21**]. The Abelian translation group provides an identification of the tangent space $T_\varphi\mathcal{H}$ at any point $\varphi\in\mathcal{H}$ with $\mathcal{H}$ itself, the isomorphism being obtained by associating with $\psi\in\mathcal{H}$ the vector $\dot\psi\in T_\varphi\mathcal{H}$ given by:

$$\dot\psi f(\varphi):=\left(\frac{d}{dt}f(\varphi+t\psi)\right)_{|t=0}\ ,$$

for any $f\in C^\infty(\mathcal{H})$.

The symplectic 2-form $\mathrm{Ker}(B)$ is given by

$$\Omega_\varphi(\dot\psi,\dot\psi')=2\,\mathrm{Im}\langle\psi,\psi'\rangle$$

with $\langle\cdot,\cdot\rangle$ denoting the Hilbert inner product on $\mathcal{H}$.

Through the identification of $\mathcal{H}$ with $T_\varphi\mathcal{H}$ a continuous vector field is just a continuous map $A\colon\mathcal{H}\to\mathcal{H}$; therefore a linear operator A on $\mathcal{H}$ is a special kind of vector field, $X_A(\psi)=A\,\psi$. On the other side, given a smooth function $a\colon\mathcal{H}\to\mathbb{R}$, its differential da_φ at $\varphi\in\mathcal{H}$ is an element of the (real) dual $\mathcal{H}'$ given by:

$$da_\varphi(\psi):=\left(\frac{d}{dt}a(\varphi+t\psi)\right)_{|t=0}\ .$$

One can see that the skew-Hermitian linear operators in $\mathcal{H}$ define Hamiltonian vector fields, the Hamiltonian function of $-i\,A$ for a self-adjoint operator A being $a(\varphi)=\frac{1}{2}\langle\varphi,A\varphi\rangle$ (see e.g. [**21**]). In this sense, the time-dependent Schrödinger equation

$$\frac{\partial\psi}{\partial t}=-i\,H\,\psi\ ,$$

plays the rôle of Hamilton's equations because it determines the integral curves of the vector field $-i\,H$.

7. Applications of Lie systems in Quantum Mechanics

Now Lie system theory applies to the case in which a t-dependent Hamiltonian can be written as a linear combination with t-dependent coefficients

$$H=\sum_{\alpha=1}^{r}b_\alpha(t)H_\alpha\ , \tag{11}$$

where the r time-independent Hamiltonians are such that the skew-Hermitian operators $i\,H_\alpha$ close on a finite-dimensional real Lie algebra, namely, there exist real numbers $c_{\alpha\beta}{}^\gamma$ such that

$$[iH_\alpha,iH_\beta]=i\sum_{\gamma=1}^{r}c_{\alpha\beta}{}^\gamma\,H_\gamma\ . \tag{12}$$

We insist, however, on the fact that this Lie algebra does not necessarily coincide with the corresponding classical one, but it is a Lie algebra extension of it and corresponds to the Lie algebra generated by the classical Hamiltonians h_α under Poisson Brackets.

As a first instance, we can deal with the very simple case of a quantum time-dependent linear potential, which has recently been studied in [**22**] and and also by Guedes [**23**] using the Lewis and Riesenfeld invariant method [**24**]. In this case the quantum Hamiltonian is

$$H(t)=-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}+q\,\epsilon(t)\,x\ .$$

Actually we see that in the quantum problem, as pointed out in [**25**], the quantum Hamiltonian H may be written as a sum $H = (1/m)\, H_1 - f(t)\, H_2$, with $H_1 = P^2/2$, $H_2 = -X$ and $f(t) = q\,\epsilon(t)$. But $-i\, H_1$ and $-i\, H_2$ are skew-Hermitian and close on a four-dimensional Lie algebra with $-i\, H_3 = -i\, P$ and $-i\, H_4 = i\, I$, which is an extension of the Heisenberg Lie algebra. Therefore, the Schrödinger equation given by the Hamiltonian H is a Lie system and we can find the time evolution operator by applying the reduction of the problem to an equation on the Lie group and using, for instance, the Wei–Norman method. We can also consider nonlinear systems. For instance the system for which [**26, 27**]

$$H = \alpha(t)\left(\frac{p^2}{2} + \frac{g}{q^2}\right) + \gamma(t)\,\frac{q^2}{2}$$

is a Hamiltonian system of Lie type with $SL(2,\mathbb{R})$ as associated group.

Another example of these systems is the one given by

$$H = \omega(t)\, J_3 + \Omega^*(t)\, J_+ + \Omega(t)\, J_-$$

which has been studied in [**28**] and [**29**], where $\omega(t)$ and $\Omega(t)$ are real and complex nonsingular functions, respectively, and $[J_3, J_\pm] = \pm J_\pm$, $[J_+, J_-] = \delta J_3$, where $\delta = 1$ for the $SU(2)$ case and $\delta = -1$ for the $SU(1,1)$ case.

Finally, we stress that the identification of Lie systems as those systems of differential equations defining the horizontal curves of a connection may be very useful in both classical and quantum physics. Actually every time we deal with a Lie system we can take advantage of the action of the group $\mathcal{G}$ of curves in G (which corresponds to the group of automorphisms of the bundle) on the set of the corresponding Lie systems, transforming the original system into other systems [**22, 25, 30**] relating in this way different systems.

For instance this is the motivation for the reduction of some time-dependent Hamiltonian systems to one of Hill–Floquet–Bloch type [**31**]. An appropriate reduction for the Lie system leads to a one-dimensional problem and then the time-dependence of the Hamiltonian is factorized as a time function. In fact the simplest example of a Lie system is one of the form

$$\dot{x} = \lambda(t)\, X(x)\ ,$$

for finite-dimensional classical systems or

$$\frac{\partial\psi}{\partial t} = -i\,\lambda(t)\, H_0\, \psi\ ,$$

with a t-independent Hamiltonian operator H_0 in the quantum case. Both cases are related with an autonomous case just by redefining the t-parameter as follows

$$\tau(t) = \int^t \lambda(\zeta)\, d\zeta\ , \tag{13}$$

the new systems being

$$\frac{dx}{d\tau} = X(x)\ , \qquad \frac{d\psi}{d\tau} = -i\, H_0\, \psi\ ,$$

respectively.

For these Hamiltonian Lie systems in which the Hamiltonian is a linear combination, with t-dependent functions as coefficients, of other Hamiltonians closing on a real Lie algebra, it is very convenient to make use of a change defined by a curve in the group (an automorphism of the bundle in the other sense) in such a way that the new Hamiltonian be of the form

$$H'(t) = \lambda(t)\, H_0\,, \tag{14}$$

with H_0 time-independent. Once this has been obtained, it suffices to redefine the time function as in (13) and then the evolution in terms of the new variable τ is τ-independent. For instance Feng and Wang [**32**] in their study of the motion of a particle in a Paul trap reduced the problem of motion under a time-dependent frequency harmonic oscillator to a time-independent one by means of somehow ad-hoc changes. The point to be remarked is that finding a method to define these changes is a particular example of the more general reduction procedure explained here: the final subgroup is $\mathbb{R}$. The reduction is based on the choice of some particular solutions in appropriate homogeneous spaces.

The method has also been used by Jing *et al.* [**31**] for studying a perturbed Landau system in two dimensions and the corresponding Berry's non-adiabatic phase in the case of periodicity of the Hamiltonian with time. The form (14) is called the Hill–Floquet–Bloch form. The particular example they have considered is a charged particle trapped in a magnetic field $B(t)$ along the z-axis, perturbed by an inverse square potential in the $x-y$ plane:

$$H(t) = H_{\rm space}(t) + H_{\rm spin}(t)\ ,$$

where

$$H_{\rm space}(t) = \frac{\boldsymbol{p}^{\,2}}{2\,m} + \frac{1}{2}\,m\,\omega^2(t)\,\boldsymbol{r}^{\,2} + \frac{\omega_c(t)}{2}\,\ell_z + \frac{\beta}{\boldsymbol{r}^{\,2}}\ ,$$

and

$$H_{\rm spin}(t) = -2\,\mu_B B(t)\, s_z\,,$$

where

$$\omega^2(t) = \frac{1}{4}\,\omega_c^2(t) \quad \text{with} \quad \omega_c(t) = e\,\frac{B(t)}{m\,c}\,.$$

The spatial and spinorial parts of the Hamiltonian commute so that the spinorial part can be treated separately. In the cited reference the spatial part of the Hamiltonian is transformed into a Hamiltonian of the form (taking $m=1$)

$$H'(t) = \lambda(t)\ \left(\frac{\boldsymbol{p}^2}{2} + \frac{1}{2}\,\Omega^2\,\boldsymbol{r}^{\,2} + \frac{\beta}{\boldsymbol{r}^{\,2}}\right)\ ,$$

where Ω is a constant, by means of the curve in the direct product $SL(2,\mathbb{R})\otimes\mathbb{R}$ given by

$$\exp(i\,\nu(t) l_z)\exp(i\,C(t)\boldsymbol{r}^2)\exp(i\,D(t)(\boldsymbol{r}\,\boldsymbol{p}+\boldsymbol{p}\,\boldsymbol{r}))\,,$$

where $\nu(t)$, $C(t)$, $D(t)$ satisfy a system of first-order differential equations, that is a particular solution of the Lie system on the corresponding homogeneous space mentioned in Theorem 1.

Acknowledgments

Financial support by research project BFM-2003-02532 is acknowledged. A.R. has been partially supported by the European Commission funding for the Human Potential Research Network "Mechanics and Symmetry in Europe" (MASIE), contract HPRN-CT-2000-00113.

References

[1] Lie S., *Vorlesungen über continuierliche Gruppen mit Geometrischen und anderen Anwendungen,* Edited and revised by G. Scheffers, Teubner, Leipzig, 1893.

[2] Cariñena J.F., *Sections along maps in Geometry and Physics,* Rend. Sem. Mat. Univ. Pol. Torino **54** (1996) 245–56.

[3] Anderson R.L., *A nonlinear superposition principle admitted by coupled Riccati equations of the projective type,* Lett. Math. Phys. **4** (1980) 1–7.

[4] Winternitz P., *Lie groups and solutions of nonlinear differential equations*, in Nonlinear Phenomena, K.B. Wolf Ed., Lecture Notes in Physics **189**, Springer-Verlag, N.Y., 1983.

[5] Olmo M.A. del, Rodríguez M.A. and Winternitz P., *Simple subgroups of simple Lie groups and nonlinear differential equations with superposition principles,* J. Math. Phys. **27** (1986) 14–23.

[6] Olmo M.A. del, Rodríguez M.A. and Winternitz P., *Superposition formulas for rectangular matrix Riccati equations,* J. Math. Phys. **28** (1987) 530–5.

[7] Cariñena J.F., Grabowski J. and Ramos A., *Reduction of time-dependent systems admitting a superposition principle,* Acta Appl. Math. **66** (2001) 67–87.

[8] Cariñena J.F., Grabowski J. and Marmo G., *Lie–Scheffers systems: a geometric approach,* Bibliopolis, Napoli, 2000.

[9] Cariñena J.F., Grabowski J. and Marmo G., *Some applications in physics of differential equation systems admitting a superposition rule*, Rep. Math. Phys. **48** (2001) 47–58.

[10] Montesinos M. and Pérez-Lorenzana A., *Exact solutions of n–level systems and gauge theories,* Phys. Rev. **60 A** (1999) 2554–7.

[11] Asorey M., Cariñena J.F. and Paramio M., *Quantum evolution as a parallel transport*, J. Math. Phys. **23** (1982) 1451–8.

[12] Wei J. and Norman E., *Lie algebraic solution of linear differential equations*, J. Math. Phys. **4** (1963) 575–81.

[13] Wei J. and Norman E., *On global representations of the solutions of linear differential equations as a product of exponentials,* Proc. Amer. Math. Soc. **15** (1964) 327–34.

[14] Cariñena J.F., Marmo G. and Nasarre J., *The nonlinear superposition principle and the Wei–Norman method*, Int. J. Mod. Phys. **A 13** (1998) 3601–27.

[15] Cariñena J.F. and Ramos A., *A new geometric approach to Lie systems and physical applications*, Acta Appl. Math. **70** (2002) 43–69.

[16] Cariñena J.F. and Ramos A., *Integrability of the Riccati equation from a group theoretical viewpoint*, Int. J. Mod. Phys. **A 14** (1999) 1935–51.

[17] Barata J.C.A. and Wreszinski W.F., *Strong coupling limit theory of two levels atoms in periodic fields*, Phys. Rev. Lett. **84** (2000) 2112–5.

[18] Infeld L. and Hull T.E., *The Factorization Method,* Rev. Mod. Phys. **23** (1951) 21–68.

[19] Cariñena J.F. and Ramos A., *Riccati equation, the factorization method and shape invariance,* Rev. Math. Phys. **12** (1999) 1279–304.

[20] Cariñena J.F., Fernández D.J. and Ramos A., *Group theoretical approach to the intertwined Hamiltonians*, Ann. Phys. (N.Y.) **292** (2001) 42–66.

[21] Boya L.J., Cariñena J.F. and Gracia-Bondía J.M., *Symplectic structure of the Aharonov–Anandan geometric phase,* Phys. Lett. **161 A** (1991) 30–4.

[22] Feng M., *Complete solution of the Schrödinger equation for the time-dependent linear potential,* Phys. Rev. **A 64** (2003) 034101.

[23] Guedes I., *Solution of the Schrödinger equation for the time-dependent linear potential,* Phys. Rev. **A 63** (2001) 034102.

[24] Lewis Jr. H.R. and Riesenfeld W.B., *An exact quantum theory of the time-dependent harmonic oscillator and of a charged particle in a time-dependent electromagnetic field*, J. Math. Phys. **10** (1969) 1458–73.

[25] Balasubramanian S., *Time-development operator method in quantum mechanics*, Am. J. Phys. **69** (2001) 508–11.

[26] Feng M., Wang K., Wu J. and Shi L., *Approximate study of the quantum statistics of two Paul trapped ions*, Phys. Lett. **A 230** (1997) 51–9.

[27] Song D.-Y., *Unitary relation between a harmonic oscillator of time-dependent frequency and a simple harmonic oscillator with or without an inverse square potential*, Phys. Rev. **A 62** (2000) 014103.

[28] Dattoli G., Richetta M. and Torre A., *Evolution of SU(2) and SU(1,1) states: A further mathematical analysis*, J. Math. Phys. **29** (1988) 2586–8.

[29] Dattoli G. and Torre A., *Cayley–Klein parameters and evolution of two- and three-level systems and squeezed states*, J. Math. Phys. **31** (1990) 236–40.

[30] Jian-Sheng Wu, Zhi-Ming Bai and Mo-Lin Ge, *The non-adiabatic Berry phase in two dimensions: the Calogero model trapped in a time-dependent external field*, J. Phys. A: Math. Gen. **32** (1999) L381–6.

[31] Hui Jing, Bing-Hao Xie and Qing-Yun Shi, *Perturbed Landau system and non-adiabatic Berry's phase in two dimensions*, Phys. Lett. **A 277** (2000) 295–8.

[32] Feng M. and Wang K., *Exact solution for the motion of a particle in a Paul trap*, Phys. Lett. **A 197** (1995) 135–8.

J. Cariñena
Departamento de Física Teórica,
Universidad de Zaragoza,
50009 Zaragoza, Spain.

e-mail: jfc@unizar.es

A. Ramos
Dipartimento di Matematica Pura ed Applicata,
Università degli Studi di Padova,
I-35131 Padova, Italy

e-mail: aramos@math.unipd.it