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Some planar multiple isoperimetric problems

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Abstract. In this survey we describe several planar multiple isoperimetric problems. We treat soap bubble problems in the plane, in the halfplane and in a corner, and specially the least-perimeter partition of a planar disk into n regions.

Introduction

Multiple isoperimetric problems are closely related to many natural situations. Among all of them, soap bubble problems or partitioning problems are good patterns in order to explain several physical phenomena. In this work we will revise several of these problems in $\mathbb{R}^2$.

An interesting isoperimetric problem consists of finding the configuration of curves enclosing n given areas with the least possible perimeter. This is the *planar soap bubble problem*, and its name comes from the soap bubbles in $\mathbb{R}^3$, whose behaviour tends to minimize the total energy, which coincides with the perimeter of the configuration in the planar case.

Partitioning problems search for the least-perimeter division of a set into n regions of prescribed areas. This problem models many situations in the Nature, as the process of division of a cell, or several honeycomb structures (see [**Th**]). We will see more applications in Section 4.

The existence of solution, that will be called *minimizer*, for this kind of problems is derived from the work in [**M**], which also describes some conditions that such minimizer must verify.

In this survey we review several multiple isoperimetric problems in $\mathbb{R}^2$. We will treat the soap bubble problem in different settings (the whole plane, the halfplane, a corner), commenting many related questions which are still open. And we will conclude with the partitioning problem inside a disk, obtaining the conditions that the minimizer must satisfy, and solving completely the problem for two and three regions.

The main difficulty in these problems is that regions may have several components. There is no way of proving that the regions have to be connected in a minimizer. Therefore there are too many possibilities of solution in each problem.

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1. Soap bubble problem in $\mathbb{R}^2$

In this section we will consider the following problem: which is the least-perimeter way of enclosing n regions of prescribed areas in the plane? Of course, the classical isoperimetric inequality gives us the solution for a single given area, obtaining a circle. This problem is only solved for two and three regions, and the solutions have certainly connected regions and their exterior is also connected (that is, there are no empty chambers).

In the general case, the minimizer must satisfy the following regularity conditions:

i) The minimizer consists of smooth curves with constant geodesic curvature.
ii) The curves meet in threes at 120-degree angles.

THEOREM 1. *(*[**FABH**]*) The least-perimeter way of enclosing two given quantities of area in* $\mathbb{R}^2$ *is by means of the standard double bubble, shown in Figure 1.*

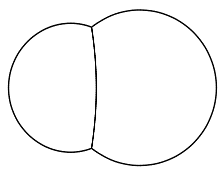


FIGURE 1. The standard double bubble

In the case $n = 3$, the problem was firstly solved by imposing connectedness of the regions ([**CHH**]), and that all the regions have the same pressure ([**D**]), obtaining the standard triple bubble. Finally, [**W**] proved the general case in 2002:

THEOREM 2. *(*[**W**]*) The least-perimeter way of enclosing three given quantities of area in* $\mathbb{R}^2$ *is by means of the standard triple bubble, shown in Figure 2.*

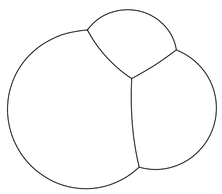


FIGURE 2. The standard triple bubble

This problem remains open for $n \geqslant 4$, and the most interesting related question is finding an argument that gives connectedness of the regions in the minimizing configuration.

Finally, pointing out that this problem has been also studied in several interesting physical papers (e.g., [**GJJF**] and [**CGFM**]).

2. Soap bubble problem in the halfplane

Now we will treat a different isoperimetric problem. Fixing a straight line in the plane, consider the same problem of the previous section, enclosing n regions of given areas using the least-possible perimeter. But now such straight line will not contribute to the total perimeter.

This is a free boundary problem, and the solution for one or two regions follows immediately from the results in Section 1. For one single area, a reflection argument gives that a *semicircle*, meeting orthogonally the straight line, is the solution (otherwise the circle would not solve the isoperimetric problem in the plane). And for two regions, an analogous proof shows that the *split standard double bubble*, that is, a standard double bubble bisected by the line, is the minimizer (see [**HLPS**]).

FIGURE 3. Minimizers in the halfplane

3. Soap bubble problem in a corner

In this section, another free boundary problem is studied. Now we will consider a planar corner of angle θ, given by two half-lines starting from the same point. Which is the least-perimeter way of enclosing n regions of given areas? Recall that we do not care about the perimeter coming from the half-lines.

For one single area, it can be checked that the solution is given by a circular arc centered at the corner.

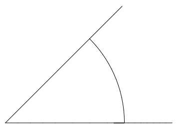


FIGURE 4. The least-perimeter enclosure of one area in a corner

In general, this problem is not completely solved. Only in the case $n = 2$, assuming that the regions and the exterior are connected, some partial results are known. In [**HLPS**] it is proved that, given two areas a_1, a_2, there exist two angles θ_0, θ_1 depending on such areas (with $\theta_0 < \theta_1$), so that the least-perimeter enclosure of two regions of area a_i in the corner is given by

i) *two concentric circles* centered at the corner, if $\theta < \theta_0$, or
ii) a *truncated standard double bubble*, shown in Figure 5, if $\theta \geqslant \theta_1$.

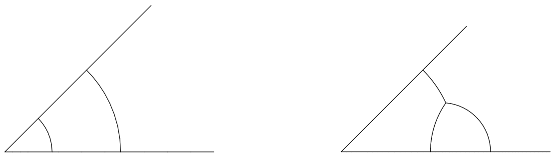


FIGURE 5. Least-perimeter configurations for two given areas in a corner:
two concentric circles and a truncated standard double bubble

So in this case, the angle θ of the corner determines the solution; and for $\theta \in [\theta_0, \theta_1)$ this question remains open. In general, without the connectedness assumption, this soap bubble problem is not solved, due to the large number of candidates that appear.

4. Least-perimeter partitions of a disk

Now we will focus on a partitioning problem modeling several natural phenomena. Consider a planar disk D, and n positive numbers $a_1, \ldots, a_n$ satisfying that $\sum_{i=1}^{n} a_i = \mathit{area}\,(D)$. Which is the least-perimeter way of dividing D into n regions R_i, each one of area a_i? We will treat this problem in detail.

As described in [**Th**, Ch. VII and VIII], this problem has several applications in physical sciences. The process of division of a cell tends to minimize the total energy, that is, the length of the membranes of the new cells appearing. And the first steps of this process will be properly explained by the problem considered here, since the cell is circular. Another situation following this model is the set of interfaces separating different inmiscible fluids in a round dish. Moreover, the solutions for this problem can be produced by blowing planar soap bubbles in a circular plate.

As in the previous problems, we obtain from [**M**] the existence of a solution:

THEOREM 3. *Given a planar disk D and $a_1, \ldots, a_n$ n positive numbers satisfying $\sum_{i=1}^{n} a_i = area(D)$, there exists a least-perimeter way of dividing the disk into n regions of areas $a_1, \ldots, a_n$, consisting of smooth curves meeting in threes in the interior of the disk, and meeting the boundary of the disk only one curve at a time.*

4.1. Regularity Conditions. In this part we will show the regularity conditions that must be satisfied by the solution.

Consider a closed disk $D \subset \mathbb{R}^2$ centered at the origin. From now on, a *graph C* will consist of a finite number of vertices and curves in D satisfying that

i) at every interior vertex (a vertex in the interior of the disk) three curves meet, and
ii) at every vertex in the boundary of the disk, only one curve meets ∂D.

Observe that this is a natural definition from Theorem 3.

We shall assume that a graph C divides the open disk into n regions R_i, which may have several components, and we shall denote by $C_{ij} \subset C$ the curve separating two adjacent regions R_i and R_j (note that this curve may not be connected), by N_{ij} the normal vector to C_{ij} pointing into R_i, and by h_{ij} the geodesic curvature of C_{ij} with respect to N_{ij}. Notice that the geodesic curvature coincides with the usual curvature of a curve since $D \subset \mathbb{R}^2$.

We will call *minimizing graph* to the least-perimeter graph dividing the disk into n regions of the given areas.

Consider a smooth variation of a graph C, $\varphi_t : C \to D$, for t small, satisfying that $\varphi_t(C \cap \partial D) \subset \partial D$. If $X = d\varphi_t/dt|_{t=0}$ is the associated infinitesimal vector field, with normal components $u_{ij} = X \cdot N_{ij}$ on C_{ij}, we can compute the derivative of the area A_i enclosed by a region R_i at $t = 0$, obtaining

$$\left.\frac{dA_i}{dt}\right|_{t=0} = -\sum_{j \in I(i)} \int_{C_{ij}} u_{ij},$$

where $I(i) = \{j \neq i; R_j \text{ touches } R_i\}$.

PROPOSITION 1. *(First variation of length) Let C be a graph and let $\varphi_t : C \to D$ be a smooth variation. Then the first derivative of the functional length of $\varphi_t(C)$ at $t = 0$ is given by*

$$\frac{dL}{dt}\bigg|_{t=0} = -\frac{1}{2} \sum_{\substack{i\in\{1,\dots,n\}\\ j\in I(i)}} \left\{ \int_{C_{ij}} h_{ij} u_{ij} + \sum_{p\in\partial C_{ij}} X(p)\cdot \nu_{ij}(p) \right\},$$

where $\nu_{ij}(p)$ is the inner conormal to C_{ij} in p.

We will say a graph is *stationary* if $\frac{dL}{dt}|_{t=0} = 0$, for any variation preserving the areas. Since we are looking for a minimum of the functional length when the areas are fixed, it is clear that a minimizing graph must be stationary.

By taking appropriate variations in Proposition 1 we can obtain the regularity conditions a stationary graph (and therefore a minimizing graph) must satisfy:

THEOREM 4. *(Regularity Conditions) A stationary graph C must satisfy the following regularity conditions:*

i) *The curvature h_{ij} is constant on C_{ij}, and therefore the edges will be circular arcs or line segments.*
ii) *The edges of C meet in threes at 120-degree angles in interior vertices.*
iii) *Balancing condition: given three edges C_{ij}, C_{jk}, C_{ki} meeting in an interior vertex, then*

$$h_{ij} + h_{jk} + h_{ki} = 0.$$

iv) *The edges of C meet ∂D orthogonally.*

4.2. Minimizing graph for two regions. From Theorem 4, we can solve the problem in the case $n = 2$. Since we are dividing the disk into two regions, there cannot be triple interior vertices in the minimizing graph, and so the only possibility for edges is to be circular arcs or line segments meeting orthogonally ∂D.

Suppose now that the minimizing graph has more than one edge. Then we can rotate one of them, about the center of the disk, until meeting another one. This movement will give another minimizing graph (since the perimeter and the areas enclosed are preserved), but creating an irregular vertex in ∂D, which yields a contradiction.

Then we have the following

THEOREM 5. *Let $C \subset D$ be a minimizing graph for two given areas. Then C consists of a circular arc or line segment meeting orthogonally ∂D.*

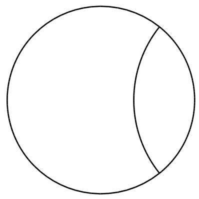


FIGURE 6. The least-perimeter partition of the disk into two given areas

4.3. Minimizing graph for three regions. Now we compute the second variation formula of length, which will give us a bound on the number of components of certain region. This bound will allow us to discuss the different possibilities for the minimizer.

PROPOSITION 2. *(Second variation of length) Given a stationary graph C and a variation $\{\varphi_t\}$ preserving the areas, the second derivative of the functional length of $\varphi_t(C)$ at $t=0$ is equal to*

$$-\frac{1}{2}\sum_{\substack{i=1,\dots,n\\ j\in I(i)}}\Bigg\{\int_{C_{ij}}(u''_{ij}+h^2_{ij}u_{ij})\,u_{ij}\;+\sum_{\substack{p\in\partial C_{ij}\\ p\in\mathrm{int}(D)}}\Big(-q_{ij}u^2_{ij}+u_{ij}\frac{\partial u_{ij}}{\partial\nu_{ij}}\Big)(p)\;+\sum_{\substack{p\in\partial C_{ij}\\ p\in\partial D}}\Big(u^2_{ij}+u_{ij}\frac{\partial u_{ij}}{\partial\nu_{ij}}\Big)(p)\Bigg\},\tag{4.1}$$

where $q_{ij}(p)=(h_{ki}+h_{kj})(p)/\sqrt{3}$, and R_k is the third region touching the vertex p.

Given a region R_i, we can define its *pressure* as a real number p_i satisfying, for any edge $C_{ij}\subset C$, that

$$h_{ij}=p_i-p_j.$$

If we consider an area-preserving variation *by stationary graphs* (we obtain stationary graphs along the deformation), then it can be checked that the second variation of length (4.1), expressed in terms of the pressures, turns

$$\frac{d^2L}{dt^2}=\sum_{\alpha}\frac{dp_\alpha}{dt}\,\frac{dA_\alpha}{dt},\tag{4.2}$$

where α labels the *components* of the graph (recall that regions may have several components).

We will say a stationary graph C is *stable* if $\frac{d^2L}{dt^2}\big|_{t=0}\geqslant 0$, for any variation preserving the areas. Then we have that a minimizing graph must be stable.

Denoting by R_1 the *region of largest pressure*, we have the following

PROPOSITION 3. *In a stable graph dividing the disk into n regions, R_1 has at most $n-1$ nonhexagonal components.*

In the case $n=3$, since hexagonal components can be discarded by using Gauss-Bonnet Theorem, this bound allows to discuss the different configurations that can appear, to finally obtain this proposition:

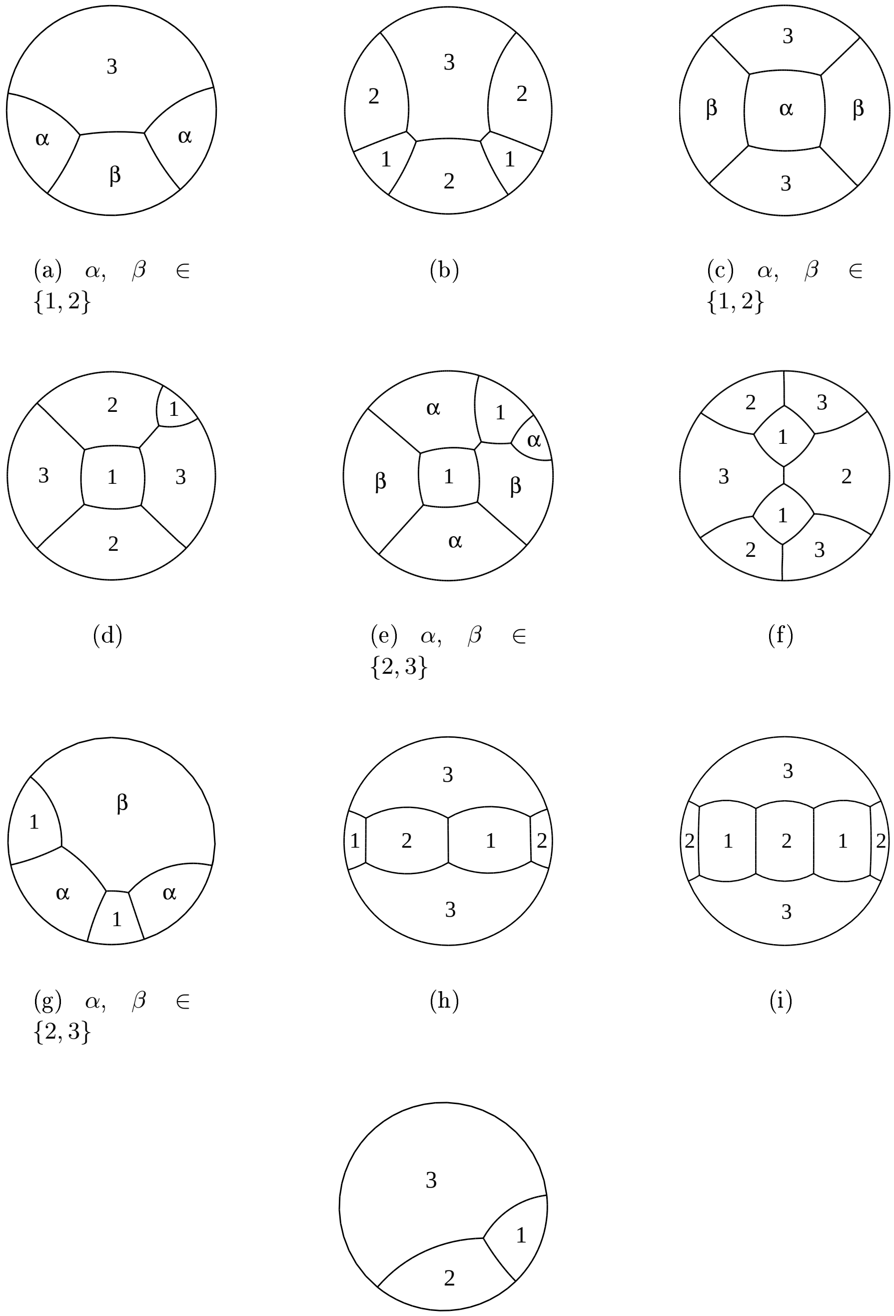
PROPOSITION 4. *Let C be a minimizing graph dividing D into three regions. Then C is one of the graphs in Figure 7.*

Configuration 7(j) will be called *standard graph*. Now we will indicate how to eliminate the nine non-standard possibilities, to conclude that the least-perimeter partition of the disk into three given areas is the standard graph.

Let us see that configurations 7(a) and 7(b) are unstable, and therefore not minimizing. The reason is that both configurations have a region with two *triangles* (components of three edges) touching ∂D.

PROPOSITION 5. *Let C be a stationary graph with a triangle Ω touching ∂D. Then there exists a variation by stationary graphs that*

(i) *increases the area of Ω,*



(a) $\alpha,\ \beta\ \in \{1,2\}$

(b)

(c) $\alpha,\ \beta\ \in \{1,2\}$

(d)

(e) $\alpha,\ \beta\ \in \{2,3\}$

(f)

(g) $\alpha,\ \beta\ \in \{2,3\}$

(h)

(i)

(j)

FIGURE 7. The ten possible configurations for minimizing graphs

(ii) *decreases the pressure of* Ω*, and does not change the rest of the pressures, and*

(iii) *leaves invariant the edges of* C *not lying in* $\partial\Omega$*.*

The main fact here is that this variation is *by stationary graphs*. Observe that, for this variation, $\frac{dp}{dt}\frac{dA}{dt} < 0$ in the triangle Ω.

Proposition 6. *Let C be a stationary graph with a region with two triangles Ω_1, Ω_2 touching ∂D. Then C is unstable.*

Sketch of the proof. Consider in each triangle Ω_i the variation given by Proposition 5. By combining both variations, it is possible to construct a new variation by stationary graphs too, preserving the areas enclosed. Computing now the second variation of length we have by Equation (4.2)

$$\left.\frac{d^2L}{dt^2}\right|_{t=0} = \sum_{\alpha} \frac{dp_\alpha}{dt}\frac{dA_\alpha}{dt} = \frac{dp_1}{dt}\frac{dA_1}{dt} + \frac{dp_2}{dt}\frac{dA_2}{dt} < 0,$$

since the only pressures changing are the ones of the triangles Ω_1 and Ω_2.

Then, by using Proposition 6, configurations 7(a) and 7(b) are unstable.

In order to prove that configurations 7(c) to 7(g) of Figure 7 are unstable, we use the following result:

Proposition 7. *(*[**HMRR**]*) Let C be a stationary graph separating the disk into three regions. If there exists a Jacobi function defined on C with at least four nodal domains, then C is unstable.*

As Jacobi function we can consider the normal components of the rotations vector field about the origin, and a nodal domain is a domain in C where such function does not vanish. Then, due to some existing symmetries we find in these configurations, we can apply Proposition 7 to obtain that they are all unstable.

Let us check now that configuration 7(h) is not minimizing. We can construct a new configuration with the same perimeter and enclosing the same areas by the geometrical transformation shown in Figure 8. Since the new configuration presents two irregular vertices, it cannot be minimizing, and so, neither configuration 7(h). A similar argument can be applied to configuration 7(i).

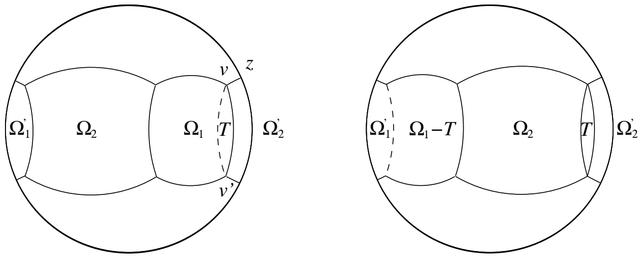


Figure 8. Geometrical transformation creating two irregular vertices

We have now discarded the nine non-standard possibilities, so we have the following theorem:

THEOREM 6. *([**CR**]) Let $C \subset D$ be a minimizing graph for three given areas. Then C is a standard graph.*

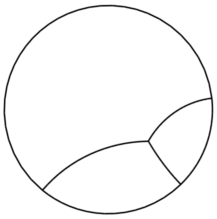
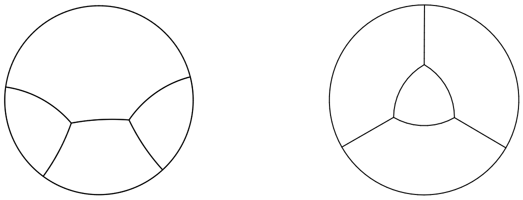


FIGURE 9. The least-perimeter partition of the disk into three given areas

Remark. For two and three regions, we also have uniqueness of solution for given areas, up to rigid motions of the disk.

4.4. Open questions. This problem remains open for $n > 3$, although the most interesting question is showing that regions must be connected in the minimizing graph.

For the case $n = 4$, a good approach to attack the problem is assuming connectedness of the regions. Under this hypothesis, there are only two possibilities for the minimizer:



(a) (b)

FIGURE 10. The two possibilities for dividing the disk into four connected regions

Consider D a planar unit disk. It is clear that by means of four appropriate radii, D will be divided into four regions of any areas $a_1, \ldots, a_4$. Hence, the perimeter of any minimizer will be less than four.

For equal areas, we have checked that the perimeter of configuration 10(b) is greater than four, and so the solution in this case is configuration 10(a). First, we see that R_1 is the interior region of configuration 10(b) (otherwise, R_4 would be completely interior and Gauss-Bonnet Theorem would give contradiction). And then, using the classical isoperimetric inequality and straightforward calculations, we obtain that the perimeter of configuration 10(a) is greater than four.

This fact and other estimates make us to conjecture that configuration 10(a) is the solution for $n = 4$.

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