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Variational principles on frame bundles

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Abstract. Variational principles on frame bundles, invariant with respect to the structure group are investigated. Explicit expressions for the first order invariant lagrangians, the Poincaré-Cartan, and the Euler-Lagrange forms are found, and the corresponding conservation laws are obtained as a consequence of the Noether's theorem. We show that the (second order) Euler-Lagrange equations for a frame field are equivalent with the system of (first order) conservation laws for this field. Analogous results are obtained for second order invariant lagrangians on frame bundles.

1. Introduction

Let FX be the frame bundle over an n-dimensional manifold X, and let J^rFX be the r-jet prolongation of FX. We shall consider J^rFX with the canonical prolongation of the right action of the general linear group $Gl_n(\mathbb{R})$ on FX. The goal of this paper is twofold. First, we describe the $Gl_n(\mathbb{R})$-invariant lagrangians on J^rFX. We also derive explicit expressions for other basic concepts of the variational theory on J^1FX and J^2FX, such as the Lepage equivalent of an invariant lagrangian, and the Euler-Lagrange form. For foundations of the variational theory in fibered space we refer to **[5]**, **[7]**, **[8]**, **[10]**, **[15]**, and the notions related to the frame bundles and invariance can be found in **[4]**, **[6]**, **[9]**, **[11]**, **[12]**, **[16]**. Second, we study the consequences of $Gl_n(\mathbb{R})$-invariance for variational problems on J^1FX and J^2FX. In particular, we discuss the corresponding Noether's currents. The generators of invariant transformations are the fundamental vector fields of the $Gl_n(\mathbb{R})$-action. Then the Noether's theorem gives us a Noether's current for each one of n^2 linearly independent fundamental vector fields. We also study the problem as to how the Noether's currents can be used to simplify the Euler-Lagrange equations. We show that in case of an invariant lagrangian defined on J^1FX, the system of n^2 second order Euler-Lagrange equations is reduced to the same number of equations of the first order.

Note that for variational problems on principal fiber bundles there are several different concepts of invariance. Muñoz and Rosado **[13]**, **[14]** study first order variational problems, invariant under diffeomorphisms of the base manifold (first order *natural* variational problems in the sense of Krupka **[7]**). Castrillón, Muñoz and

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Ratiu consider invariance of the first order lagrangians in the bundle of connections of a principal fiber bundle P over a manifold M, induced by automorphisms of P [**3**], and in J^1T^*M [**2**].

2. Invariant Lagrange structures

In this section we recall basic notions of the theory of invariant lagrangians, and introduce our notation. For a more complete discussion we refer to [**1**].

If Y is a fibered manifold over an n-dimensional manifold X, of dimension $n+m$, we denote by J^rY the *r-jet prolongation* of Y, and $\pi^{r,s}: J^rY \to J^sY$, $\pi^r: J^rY \to X$ are the *canonical jet projections.* The *r-jet* of a section γ of Y at a point $x \in X$, is denoted $J^r_x\gamma$; and $x \to J^r\gamma(x) = J^r_x\gamma$ is the *r-jet prolongation* of γ. Any *fibered chart* (V,ψ), $\psi = (x^i, y^\sigma)$, on Y, where $1 \leqslant i \leqslant n$, $1 \leqslant \sigma \leqslant m$, induces the *associated charts* on X and on J^rY, (U,φ), $\varphi = (x^i)$, and (V^r, ψ^r), $\psi^r = (x^i, y^\sigma, y^\sigma_{j_1}, y^\sigma_{j_1j_2}, \dots, y^\sigma_{j_1j_2\dots j_r})$, respectively; here $V^r = (\pi^{r,0})^{-1}(V)$, and $U = \pi(V)$. Recall that the *formal derivative operator* is defined by

$$d_i = \frac{\partial}{\partial x^i} + y^\sigma_i \frac{\partial}{\partial y^\sigma} + y^\sigma_{i_1 i} \frac{\partial}{\partial y^\sigma_{i_1}} + \dots + y^\sigma_{i_1 i_2 \dots i_r i} \frac{\partial}{\partial y^\sigma_{i_1 i_2 \dots i_r}}.$$

For any open set $W \subset Y$, Ω^r_0W denotes the ring of smooth functions on W^r. The Ω^r_0W-module of differential q-forms on W^r is denoted by Ω^r_qW, and the exterior algebra of forms on W^r is denoted by Ω^rW. The module of $\pi^{r,0}$-horizontal (π^r-horizontal) q-forms is denoted by $\Omega^r_{q,Y}W$ ($\Omega^r_{q,X}W$, respectively).

The *horizontalization* is the exterior algebra morphism $h: \Omega^rW \to \Omega^{r+1}W$, defined, in any fibered chart (V,ψ), $\psi = (x^i, y^\sigma)$, by

$$hf = f \circ \pi^{r+1,r}, \quad hdx^i = dx^i, \quad hdy^\sigma_{j_1j_2\dots j_p} = y^\sigma_{j_1j_2\dots j_pk}dx^k,$$

where $f: W^r \to \mathbb{R}$ is a function, and $0 \leqslant p \leqslant r$. A form $\eta \in \Omega^r_kW$ is *contact*, if $h\eta = 0$. For any fibered chart (V,ψ), $\psi = (x^i, y^\sigma)$, the 1-forms

$$\omega^\sigma_{j_1j_2\dots j_p} = dy^\sigma_{j_1j_2\dots j_p} - y^\sigma_{j_1j_2\dots j_pk}dx^k,$$

where $1 \leqslant p \leqslant r-1$, are examples of contact 1-forms. Note that these forms define a basis of 1-forms on V^r, $(dx^i, \omega^\sigma_{j_1j_2\dots j_p}, dy^\sigma_{j_1j_2\dots j_r})$.

It is known that a form $\eta \in \Omega^r_kW$ has a unique decomposition

$$(\pi^{r+1,r})^*\eta = h\eta + p_1\eta + p_2\eta + \dots + p_k\eta, \tag{1}$$

such that $p_i\eta$ contains, in any fibered chart, exactly i exterior factors $\omega^\sigma_{j_1j_2\dots j_l}$, $1 \leqslant l \leqslant r$.

$h\eta$ ($p_i\eta$) is the *horizontal* (*i-th contact*) *component* of η. The decomposition (1) is *invariant*, and is called the *canonical decomposition* of η. η is *π^r-horizontal* if and only if $(\pi^{r+1,r})^*\eta = h\eta$. We say that η is *k-contact*, if $(\pi^{r+1,r})^*\eta = p_k\eta$; in this case k is the *order of contactness* of η. We say that the order of contactness of η is $\leqslant l$, if $p_{l+1}\eta = p_{l+2}\eta = \dots = p_k\eta = 0$.

A *lagrangian* (of *order r*) for Y is any π^r-horizontal n-form on some W^r. A differential form $\rho \in \Omega^s_nW$, where $n = \dim X$, is called a *Lepage form*, if $p_1d\rho$ is $\pi^{s+1,0}$-horizontal, i.e. $p_1d\rho \in \Omega^{s+1}_{n+1,Y}W$. A Lepage form ρ is a *Lepage equivalent* of a lagrangian $\lambda \in \Omega^r_{n,X}W$, if $h\rho = \lambda$ (possibly up to a jet projection).

In a fibered chart (V,ψ), $\psi=(x^i,y^\sigma)$, denote

$$\omega_0 = dx^1\wedge dx^2\wedge\ldots\wedge dx^n,\quad \omega_{i_1i_2\ldots i_k} = i_{\partial/\partial x^{i_k}}\ldots i_{\partial/\partial x^{i_2}} i_{\partial/\partial x^{i_1}}\omega_0.$$

In this fibered chart, a lagrangian, defined on $V^r=(\pi^{r,0})^{-1}(V)$, has an expression

$$\lambda = \mathcal{L}\omega_0, \tag{2}$$

where $\mathcal{L}: V^r\to\mathbb{R}$ is the *Lagrange function* associated with λ and (V,ψ). A pair (Y,λ), consisting of a fibered manifold Y and a lagrangian λ of order r for Y is called a *Lagrange structure* (of order r).

We give three examples of Lepage equivalents:

(1) Every first order lagrangian $\lambda\in\Omega^1_{n,X}W$ has a unique Lepage equivalent $\Theta_\lambda\in\Omega^1_{n,Y}W$ whose order of contactness is $\leqslant 1$. If λ is expressed by (2), then

$$\Theta_\lambda = \mathcal{L}\omega_0 + \frac{\partial\mathcal{L}}{\partial y^\sigma_i}\omega^\sigma\wedge\omega_i. \tag{3}$$

Θ_λ is the *Poincaré-Cartan equivalent* of λ, or the *Poincaré-Cartan form.*

(2) The *fundamental Lepage equivalent* $\Phi_\lambda\in\Omega^1_{n,Y}W$ of $\lambda\in\Omega^1_{n,X}W$ is given by

$$\Phi_\lambda = \sum_{k=0}^{n}\left(\frac{1}{k!}\right)^2\frac{\partial^k\mathcal{L}}{\partial y^{\sigma_1}_{j_1}\partial y^{\sigma_2}_{j_2}\ldots\partial y^{\sigma_k}_{j_k}}\omega^{\sigma_1}\wedge\omega^{\sigma_2}\wedge\ldots\wedge\omega^{\sigma_k}\wedge\omega_{j_1j_2\ldots j_k}.$$

Φ_λ has the following remarkable properties: (a) $d\Phi_\lambda=0$ if and only if $E_\lambda=0$, and (b) $\lambda=h\eta$ for some $\eta\in\Omega^0_nW$ if and only if E_λ is $\pi^{2,1}$-projectable [**9**], [**10**].

(3) Formula

$$\Theta_\lambda = \mathcal{L}\omega_0 + \left(\frac{\partial\mathcal{L}}{\partial y^\sigma_i} - d_p\frac{\partial\mathcal{L}}{\partial y^\sigma_{pi}}\right)\omega^\sigma\wedge\omega_i + \frac{\partial\mathcal{L}}{\partial y^\sigma_{ji}}\omega^\sigma_j\wedge\omega_i \tag{4}$$

generalizes the Poincaré-Cartan form to *second order* lagrangians $\lambda\in\Omega^2_{n,X}W$ [**9**].

If ρ is a Lepage equivalent of a lagrangian $\lambda\in\Omega^r_{n,X}W$, $\lambda=\mathcal{L}\omega_0$, then by a direct calculation $p_1d\rho = E_\sigma(\mathcal{L})\omega^\sigma\wedge\omega_0$, where

$$E_\sigma(\mathcal{L}) = \sum_{k=0}^{r}(-1)^k d_{i_1}d_{i_2}\ldots d_{i_k}\frac{\partial\mathcal{L}}{\partial y^\sigma_{i_1i_2\ldots i_k}}$$

are the *Euler-Lagrange expressions.* The $(n+1)$-form

$$E_\lambda = p_1 d\rho$$

is the *Euler-Lagrange form* associated with λ.

By an *automorphism* of Y we mean a diffeomorphism $\alpha: W\to Y$, where $W\subset Y$ is an open set, such that there exists a diffeomorphism $\alpha_0:\pi(W)\to X$ such that $\pi\alpha=\alpha_0\pi$. If α_0 exists, it is unique, and is called the *π-projection* of α. The *r-jet prolongation* of α is an automorphism $J^r\alpha: W^r\to J^rY$ of J^rY, defined by

$$J^r\alpha(J^r_x\gamma) = J^r_{\alpha_0(x)}(\alpha\gamma\alpha_0^{-1}).$$

If ξ is a π-projectable vector field on Y, and α_t is the local one-parameter group of ξ with projection $\alpha_{(0)t}$, we define the *r-jet prolongation* of ξ to be the vector field $J^r\xi$ on J^rY whose local one-parameter group is $J^r_{\alpha_t}$. Thus,

$$J^r\xi(J^r_x\gamma) = \left\{\frac{d}{dt}J^r_{\alpha_{(0)t}(x)}(\alpha_t\gamma\alpha^{-1}_{(0)t})\right\}_0.$$

The chart expression for $J^r\xi$ can be found in [**8**] or [**9**].

We now compute the Lie derivative $\partial_{J^r\xi}\lambda$. Choose to this purpose a Lepage equivalent ρ of λ, and denote by s the *order* of ρ. Since $\lambda = h\rho$, or, which is the same, $J^r\gamma^*\lambda = J^s\gamma^*\rho$ for all sections γ, we obtain

$$J^r\gamma^*\partial_{J^r\xi}\lambda = J^s\gamma^*\partial_{J^s\xi}\rho = J^s\gamma^*(i_{J^s\xi}d\rho + di_{J^s\xi}\rho).$$

Omitting γ and using the Euler-Lagrange form we get

$$\partial_{J^r\xi}\lambda = hi_{J^{s+1}\xi}E_\lambda + hdi_{J^s\xi}\rho.$$

This is the *differential first variation formula*; the first term on the right is the *Euler-Lagrange term*, and the second one is the *boundary term*.

An automorphism $\alpha : W \to Y$ of Y is said to be an *invariant transformation* of a form $\eta \in \Omega_p^s W$, if

$$J^s\alpha^*\eta = \eta.$$

We say that a π-projectable vector field ξ is the *generator* of invariant transformations of η, if

$$\partial_{J^s\xi}\eta = 0.$$

This definition includes the notions of invariance of *lagrangians*, and the *Euler-Lagrange forms*. The following simple consequence of the first variation formula is known as the *Noether's theorem*. Let $\lambda \in \Omega^r_{n,X}W$ be a lagrangian, let $\rho \in \Omega^s_n W$ be a Lepage equivalent of λ, and let γ be an extremal. Then for any generator ξ of invariant transformations of λ,

$$dJ^s\gamma^*i_{J^s\xi}\rho = 0.$$

3. Invariant differential forms on frame bundles

Let X be an n-dimensional smooth manifold. A *frame* at a point $x \in X$ is a pair $\Xi = (x, \zeta)$, where $\zeta = (\zeta_1, \zeta_2, \dots, \zeta_n)$ is a basis of the tangent space T_xX. The set FX of all frames has the structure of a right principal $Gl_n(\mathbb{R})$-bundle over X, of dimension $n(n+1)$. We denote by μ the projection of FX.

Any chart (U, φ), $\varphi = (x^i)$, on X defines the *associated chart* on FX as the pair (V, ψ), $\psi = (x^i, x^i_j)$, as follows. We set $V = \mu^{-1}(U)$, and for every $\Xi \in V$, $\Xi = (x, (\zeta_1, \zeta_2, \dots, \zeta_n))$,

$$x^i(\Xi) = x^i(x), \quad \zeta_j = x^i_j(\Xi)\left(\frac{\partial}{\partial x^i}\right)_x.$$

Obviously, $\det x^i_j \neq 0$ at every frame $\Xi \in V$. We define the inverse matrix y^j_k by setting

$$x^i_j y^j_k = \delta^i_k.$$

The corresponding associated chart on J^rFX is then denoted by (V^r, ψ^r), where $V^r = (\mu^r)^{-1}(U)$, and $\psi^r = (x^i, x^i_j, x^i_{j,k_1}, x^i_{j,k_1k_2}, \dots, x^i_{j,k_1k_2\dots k_r})$.

The right action $(\Xi, A) \to \Xi \cdot A = R_A(\Xi)$ of $Gl_n(\mathbb{R})$ on FX is expressed, in a chart (U, φ), $\varphi = (x^i)$, on X, by the equations

$$\bar{x}^i = x^i \circ R_A = x^i, \quad \bar{x}^i_j = x^i_j \circ R_A = x^i_k a^k_j. \tag{5}$$

In this formula, a^k_j are the canonical coordinates on $Gl_n(\mathbb{R})$. This action can be canonically prolonged to J^rFX. Let $A \in Gl_n(\mathbb{R})$, $J^r_x\gamma \in J^rFX$. We define

$$J^r_x\gamma \cdot A = J^r_x(R_A \circ \gamma).$$

In the associated charts on FX and J^rFX,

$$\bar{x}^i = x^i \circ R_A = x^i, \quad \bar{x}^i_{j,k_1k_2\dots k_m} = x^i_{l,k_1k_2\dots k_m} a^l_j, \tag{6}$$

where $0 \leqslant m \leqslant r$.

Eliminating the group variables from equations (5), we obtain

$$a^l_j = y^l_i \bar{x}^i_j, \tag{7}$$

hence

$$\bar{y}^j_p \bar{x}^i_{j,k_1k_2\dots k_m} = y^l_p x^i_{l,k_1k_2\dots k_m}$$

for $1 \leqslant m \leqslant r$. The functions

$$\Gamma^i_{k_1k_2\dots k_m p} = -y^l_p x^i_{l,k_1k_2\dots k_m} \tag{8}$$

are therefore invariant with respect to the group action (6). Thus, we have the following equations of the $Gl_n(\mathbb{R})$-orbits in J^rFX

$$\Gamma^i_{k_1k_2\dots k_m p} = c^i_{k_1k_2\dots k_m p},$$

where $c^i_{k_1k_2\dots k_m p} \in \mathbb{R}$.

Setting $\Psi^r = (x^i, x^i_p, \Gamma^i_{k_1p}, \Gamma^i_{k_1k_2p}, \dots, \Gamma^i_{k_1k_2\dots k_r p})$, we obtain a chart (V^r, Ψ^r) on J^rFX. Indeed, formulas (8) admit the inverse transformation

$$x^i = x^i, \quad x^i_j = x^i_j, \quad x^i_{j,k_1k_2\dots k_m} = -x^p_j \Gamma^i_{k_1k_2\dots k_m p}.$$

The chart (V^r, Ψ^r) on J^rFX is said to be $Gl_n(\mathbb{R})$-*adapted*.

We denote

$$C^rX = J^rFX/Gl_n(\mathbb{R}).$$

Using the canonical quotient projections of J^kFX onto C^kX we have the commutative diagram

$$\begin{array}{ccccccccccc}
J^rFX & \to & J^{r-1}FX & \to & \dots & \to & J^2FX & \to & J^1FX & \to & FX \\
\downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \downarrow \\
C^rX & \to & C^{r-1}X & \to & \dots & \to & C^2X & \to & C^1X & \to & X
\end{array}$$

Covering J^rFX with the $Gl_n(\mathbb{R})$-adapted charts, we can easily verify the following assertion.

LEMMA 1. *J^rFX is a right principal $Gl_n(\mathbb{R})$-bundle over C^rX.*

In particular, we see that a function on J^rFX is $Gl_n(\mathbb{R})$-invariant if and only if it depends, locally, on the coordinates $(x^i, \Gamma^i_{k_1p}, \Gamma^i_{k_1k_2p}, \dots, \Gamma^i_{k_1k_2\dots k_r p})$ on C^rX only.

We now determine the fundamental vector fields on J^rFX. Let $gl_n(\mathbb{R})$ be the Lie algebra of $Gl_n(\mathbb{R})$. Recall that the fundamental vector field ξ on FX, associated with a vector $\xi_0 \in gl_n(\mathbb{R})$, is defined by the formula $\xi(\Xi) = T_e R_\Xi \cdot \xi_0$, where e is the unit of $Gl_n(\mathbb{R})$ (see e.g. [**11**]). Expressing ξ_0 in the form

$$\xi_0 = \xi^i_j \left(\frac{\partial}{\partial a^i_j} \right)_e,$$

we get

LEMMA 2. (a) *The fundamental vector field ξ on FX, associated with ξ_0, has an expression*

$$\xi = \xi^i_j x^p_i \frac{\partial}{\partial x^p_j}.$$

(b) *The r-jet prolongation $J^r\xi$ coincides with the fundamental vector field on J^rFX, associated with ξ_0, and has an expression*

$$J^r\xi = \xi^i_j \left(x^p_i \frac{\partial}{\partial x^p_j} + x^p_{i,k_1} \frac{\partial}{\partial x^p_{j,k_1}} + x^p_{i,k_1k_2} \frac{\partial}{\partial x^p_{j,k_1k_2}} + \ldots + x^p_{i,k_1k_2\ldots k_r} \frac{\partial}{\partial x^p_{j,k_1k_2\ldots k_r}} \right).$$

LEMMA 3. *In the $Gl_n(\mathbb{R})$-adapted chart, the fundamental vector field on J^rFX, associated with ξ_0, has an expression*

$$J^r\xi = \xi^i_j x^p_i \left(\frac{\partial}{\partial x^p_j} + \sum_{s=1}^{r} \sum_{k_1 \leqslant k_2 \leqslant \ldots \leqslant k_s} y^j_l \Gamma^t_{k_1k_2\ldots k_s p} \frac{\partial}{\partial \Gamma^t_{k_1k_2\ldots k_s l}} \right).$$

We now derive a criterion for a form on J^rFX to be $Gl_n(\mathbb{R})$-invariant.

THEOREM 1. *A k-form η on J^rFX is $Gl_n(\mathbb{R})$-invariant if and only if in any chart on X,*

$$\begin{aligned} \eta = {} & \Delta_0 + y^{s_1}_{p_1} dx^{r_1}_{s_1} \wedge \Delta^{p_1}_{r_1} + y^{s_1}_{p_1} y^{s_2}_{p_2} dx^{r_1}_{s_1} \wedge dx^{r_2}_{s_2} \wedge \Delta^{p_1p_2}_{r_1r_2} \\ & + \ldots + y^{s_1}_{p_1} y^{s_2}_{p_2} \ldots y^{s_k}_{p_k} dx^{r_1}_{s_1} \wedge dx^{r_2}_{s_2} \ldots \wedge dx^{r_k}_{s_k} \wedge \Delta^{p_1p_2\ldots p_k}_{r_1r_2\ldots r_k}, \end{aligned} \tag{9}$$

where Δ_0, $\Delta^{p_1}_{r_1}$, $\Delta^{p_1p_2}_{r_1r_2}$, $\ldots$, $\Delta^{p_1p_2\ldots p_k}_{r_1r_2\ldots r_k}$ are arbitrary forms defined on C^rX.

PROOF. In what follows we use a chart (U, φ), $\varphi = (x^i)$, on X, the associated charts on FX, and J^rFX, and the $Gl_n(\mathbb{R})$-adapted chart on J^rFX. The right action of $Gl_n(\mathbb{R})$ is defined by (6), and the functions $\Gamma^i_{k_1k_2\ldots k_m p}$ (8) are $Gl_n(\mathbb{R})$-invariant.

Let η be a $Gl_n(\mathbb{R})$-invariant k-form. η has an expression

$$\begin{aligned} \eta = {} & \eta_0 + dx^{r_1}_{s_1} \wedge \eta^{s_1}_{r_1} + dx^{r_1}_{s_1} \wedge dx^{r_2}_{s_2} \wedge \eta^{s_1s_2}_{r_1r_2} \\ & + \ldots + dx^{r_1}_{s_1} \wedge dx^{r_2}_{s_2} \wedge \ldots \wedge dx^{r_k}_{s_k} \wedge \eta^{s_1s_2\ldots s_k}_{r_1r_2\ldots r_k}, \end{aligned}$$

where the forms η_0, $\eta^{s_1}_{r_1}$, $\eta^{s_1s_2}_{r_1r_2}$, $\ldots$, $\eta^{s_1s_2\ldots s_k}_{r_1r_2\ldots r_k}$ do not contain any exterior factor dx^r_s. Using the invariance condition $(J^rR_A)^*\eta = \eta$, where A runs through $Gl_n(\mathbb{R})$, and formula (7), we obtain, for every m, $1 \leqslant m \leqslant k$,

$$\eta_0 = \bar{\eta}_0, \quad x^{p_1}_{s_1} x^{p_2}_{s_2} \ldots x^{p_m}_{s_m} \eta^{s_1s_2\ldots s_m}_{r_1r_2\ldots r_m} = \bar{x}^{p_1}_{s_1} \bar{x}^{p_2}_{s_2} \ldots \bar{x}^{p_m}_{s_m} \bar{\eta}^{s_1s_2\ldots s_m}_{r_1r_2\ldots r_m}.$$

Consequently, the forms

$$\Delta_0 = \eta_0, \quad \Delta^{p_1p_2\ldots p_m}_{r_1r_2\ldots r_m} = x^{p_1}_{s_1} x^{p_2}_{s_2} \ldots x^{p_m}_{s_m} \eta^{s_1s_2\ldots s_m}_{r_1r_2\ldots r_m}$$

do not contain any exterior factor dx^r_s, and their components are $Gl_n(\mathbb{R})$-invariant functions. Then by Lemma 1, these functions depend on x^i, $\Gamma^i_{k_1p}$, $\Gamma^i_{k_1k_2p}$, $\ldots$, $\Gamma^i_{k_1k_2\ldots k_rp}$ only. In other words, the forms Δ_0, $\Delta^{p_1p_2\ldots p_m}_{r_1r_2\ldots r_m}$ are defined on the quotient space C^rX.

Conversely, suppose that we have a form η (9) in which Δ_0, $\Delta^{p_1}_{r_1}$, $\Delta^{p_1p_2}_{r_1r_2}$, $\ldots$, $\Delta^{p_1p_2\ldots p_k}_{r_1r_2\ldots r_k}$ are defined on C^rX. Then, in particular,

$$(J^rR_A)^*\Delta_0 = \Delta_0, \quad (J^rR_A)^*\Delta^{p_1p_2\ldots p_m}_{r_1r_2\ldots r_m} = \Delta^{p_1p_2\ldots p_m}_{r_1r_2\ldots r_m}.$$

Since

$$(J^r R_A)^*(y_p^s dx_s^r) = (y_p^s \circ J^r R_A) d(x_s^r \circ J^r R_A)$$
$$= \bar{y}_p^s d(x_l^r a_s^l) = \bar{y}_p^s a_s^l dx_l^r = \bar{y}_p^s y_m^l \bar{x}_s^m dx_l^r = y_p^l dx_l^r,$$

η satisfies $(J^r R_A)^* \eta = \eta$. □

4. Invariant variational principles on frame bundles

Using the results of Section 3, we determine in this section $Gl_n(\mathbb{R})$-invariant lagrangians on $J^r FX$. Then we give explicit expressions of the Lepage forms Θ_λ, the Euler-Lagrange forms, and the Noether's currents for invariant lagrangians on $J^1 FX$ and $J^2 FX$. To derive these expressions, we use the $Gl_n(\mathbb{R})$-adapted charts. Their $Gl_n(\mathbb{R})$-invariance will then follow from Theorem 1.

Let $\lambda \in \Omega^r_{n,X} FX$ be a lagrangian. In a chart (U, φ), $\varphi = (x^i)$, on X, and in the $Gl_n(\mathbb{R})$-adapted chart (V^r, Ψ^r), $\Psi^r = (x^i, x_p^i, \Gamma^i_{k_1 p}, \Gamma^i_{k_1 k_2 p}, \ldots, \Gamma^i_{k_1 k_2 \ldots k_r p})$, on $J^r FX$,

$$\lambda = \mathcal{L} \omega_0,$$

where $\mathcal{L} : V^r \to \mathbb{R}$ is the associated Lagrange function.

Theorem 2. *A lagrangian λ on $J^r FX$ is $Gl_n(\mathbb{R})$-invariant if and only if in any $Gl_n(\mathbb{R})$-adapted chart (V^r, Ψ^r), the Lagrange function $\mathcal{L}$ depends on*

$$x^i, \Gamma^i_{k_1 p}, \Gamma^i_{k_1 k_2 p}, \ldots, \Gamma^i_{k_1 k_2 \ldots k_r p}$$

only.

Proof. According to Theorem 1, λ must be projectable to $C^r X$. □

We now restrict our discussion to lagrangians of order 1 and 2. We denote by $\Psi_{\lambda,\xi}$ the Noether's current associated with the Lepage form Θ_λ (3), (4), and a vector field ξ, i.e.,

$$\Psi_{\lambda,\xi} = i_{J^r \xi} \Theta_\lambda.$$

Note that we have the following expressions for the contact forms $\omega_j^i, \omega_{j,k}^i$;

$$\omega_j^i = dx_j^i - x_{j,m}^i dx^m = dx_j^i + x_j^p \Gamma^i_{mp} dx^m,$$
$$\omega_{j,k}^i = dx_{j,k}^i - x_{j,km}^i dx^m = dx_{j,k}^i + x_j^p \Gamma^i_{kmp} dx^m.$$

The chart expression of a fundamental vector field ξ are given in Lemma 3.

The following assertion is needed for the proofs.

Lemma 4. *If f is a $Gl_n(\mathbb{R})$-invariant function on $J^r FX$, then in any $Gl_n(\mathbb{R})$-adapted chart (V^{r+1}, Ψ^{r+1}), $\Psi^{r+1} = (x^i, x_p^i, \Gamma^i_{k_1 p}, \Gamma^i_{k_1 k_2 p}, \ldots, \Gamma^i_{k_1 k_2 \ldots k_r k_{r+1} p})$,*

$$d_p f = \frac{\partial f}{\partial x^p} + \sum_{s=1}^{r} \sum_{k_1 \leqslant k_2 \leqslant \ldots \leqslant k_s} \frac{\partial f}{\partial \Gamma^i_{k_1 k_2 \ldots k_s j}} (\Gamma^i_{k_1 k_2 \ldots k_s p j} + \Gamma^i_{k_1 k_2 \ldots k_s q} \Gamma^q_{p j}).$$

The following results can be obtained by a direct computation.

LEMMA 5. *Let $\lambda \in \Omega^1_{n,X}FX$ be a $Gl_n(\mathbb{R})$-invariant lagrangian, and let ξ be a fundamental vector field on FX. Then in a $Gl_n(\mathbb{R})$-adapted chart,*

$$\Theta_\lambda = \left(\mathcal{L} - \Gamma^i_{kl}\frac{\partial \mathcal{L}}{\partial \Gamma^i_{kl}}\right)\omega_0 - \frac{\partial \mathcal{L}}{\partial \Gamma^i_{kl}} y^j_l dx^i_j \wedge \omega_k,$$

$$E_\lambda = \left(-\Gamma^q_{pi}\frac{\partial \mathcal{L}}{\partial \Gamma^q_{pl}} + \Gamma^l_{pq}\frac{\partial \mathcal{L}}{\partial \Gamma^i_{pq}} + d_p\frac{\partial \mathcal{L}}{\partial \Gamma^i_{pl}}\right) y^j_l dx^i_j \wedge \omega_0,$$

$$\Psi_{\lambda,\xi} = -\xi^m_j y^j_l x^i_m \frac{\partial \mathcal{L}}{\partial \Gamma^i_{kl}}\omega_k.$$

LEMMA 6. *Let $\lambda \in \Omega^2_{n,X}FX$ be a $Gl_n(\mathbb{R})$-invariant lagrangian, and let ξ be a fundamental vector field on FX. Then in a $Gl_n(\mathbb{R})$-adapted chart,*

$$\Theta_\lambda = \left(\mathcal{L} - \Gamma^i_{kl}\left(\frac{\partial \mathcal{L}}{\partial \Gamma^i_{kl}} - \Gamma^l_{pq}\frac{\partial \mathcal{L}}{\partial \Gamma^i_{pkq}} - d_p\frac{\partial \mathcal{L}}{\partial \Gamma^i_{pkl}}\right) - \Gamma^i_{lkq}\frac{\partial \mathcal{L}}{\partial \Gamma^i_{lkq}}\right)\omega_0$$
$$+\frac{\partial \mathcal{L}}{\partial \Gamma^i_{lkq}} d\Gamma^i_{lq} \wedge \omega_k + \left(\Gamma^q_{pi}\frac{\partial \mathcal{L}}{\partial \Gamma^q_{pkl}} + \Gamma^l_{pq}\frac{\partial \mathcal{L}}{\partial \Gamma^i_{pkq}} + d_p\frac{\partial \mathcal{L}}{\partial \Gamma^i_{pkl}} - \frac{\partial \mathcal{L}}{\partial \Gamma^i_{kl}}\right) y^j_l dx^i_j \wedge \omega_k,$$

$$E_\lambda = \left(-\Gamma^p_{qi}\frac{\partial \mathcal{L}}{\partial \Gamma^p_{ql}} - \Gamma^p_{qmi}\frac{\partial \mathcal{L}}{\partial \Gamma^p_{qml}} + \Gamma^l_{pq}\frac{\partial \mathcal{L}}{\partial \Gamma^i_{pq}} + d_p\frac{\partial \mathcal{L}}{\partial \Gamma^i_{pl}} - \Gamma^l_{pqm}\frac{\partial \mathcal{L}}{\partial \Gamma^i_{pqm}}\right.$$
$$\left.-2\Gamma^l_{pt}\left(\Gamma^t_{qm}\frac{\partial \mathcal{L}}{\partial \Gamma^i_{pqm}} + d_q\frac{\partial \mathcal{L}}{\partial \Gamma^i_{pqt}}\right) - d_p d_q\frac{\partial \mathcal{L}}{\partial \Gamma^i_{pql}}\right) y^j_l dx^i_j \wedge \omega_0,$$

$$\Psi_{\lambda,\xi} = \xi^m_j y^j_l x^i_m\left(-\frac{\partial \mathcal{L}}{\partial \Gamma^i_{kl}} + 2\Gamma^q_{pi}\frac{\partial \mathcal{L}}{\partial \Gamma^q_{pkl}} + \Gamma^l_{pq}\frac{\partial \mathcal{L}}{\partial \Gamma^i_{pkq}} + d_p\frac{\partial \mathcal{L}}{\partial \Gamma^i_{pkl}}\right)\omega_k.$$

5. Reduction of the Euler-Lagrange equations

In this section we discuss consequences of $Gl_n(\mathbb{R})$-invariance of the first, and second order lagrangians on the frame bundles for the Euler-Lagrange equations. Our basic tool is the first variation formula (Section 2).

Let X be an n-dimensional manifold, let FX be the bundle of frames over X, and let μ be the bundle projection. Suppose that we have a lagrangian $\lambda \in \Omega^1_{n,X}FX$ and a *μ-vertical* vector field ξ on FX. Then in our standard notation

$$\partial_{J^1\xi}\lambda = i_{J^2\xi}E_\lambda + hdi_{J^1\xi}\Theta_\lambda, \tag{10}$$

where Θ_λ is the Poincaré-Cartan equivalent of λ.

THEOREM 3. *Let $\lambda \in \Omega^1_{n,X}FX$ be a $Gl_n(\mathbb{R})$-invariant lagrangian, and let γ be a section of FX. The following conditions are equivalent:*

(a) *γ satisfies the Euler-Lagrange equations,*

$$E_\lambda \circ J^2\gamma = 0.$$

(b) *For any fundamental vector field ξ on FX, γ satisfies the conservation law*

$$d(J^1\gamma)^* i_{J^1\xi}\Theta_\lambda = 0.$$

(c) *For any chart* (U, φ), $\varphi = (x^i)$, *on* X, *and all* j, k, *there exist* $(n-2)$*-forms* η_k^j *such that*

$$J^1\gamma^* \left(y_l^j x_k^i \frac{\partial \mathcal{L}}{\partial \Gamma^i_{ml}} \omega_m - d\eta_k^j \right) = 0.$$

Proof. By hypothesis, for any fundamental vector field ξ on FX, $\partial_{J^1\xi}\lambda = 0$. Consequently, since ξ is always μ-vertical, the first variation formula (10) reduces to

$$i_{J^2\xi} E_\lambda + hd\Psi_{\lambda,\xi} = 0. \tag{11}$$

We can write this identity in a chart (U, φ), $\varphi = (x^i)$, on X. ξ is of the form

$$\xi = \xi_j^k x_k^i \frac{\partial}{\partial x_j^i},$$

so we have, according to Lemma 5,

$$i_{J^2\xi} E_\lambda = E_i^j(\mathcal{L}) \xi_j^k x_k^i \omega_0,$$

where

$$E_i^j(\mathcal{L}) = \left(-\Gamma^p_{qi} \frac{\partial \mathcal{L}}{\partial \Gamma^p_{ql}} + \Gamma^l_{pq} \frac{\partial \mathcal{L}}{\partial \Gamma^i_{pq}} + \frac{\partial^2 \mathcal{L}}{\partial x^p \partial \Gamma^i_{pl}} + (\Gamma^k_{mps} + \Gamma^k_{mq}\Gamma^q_{ps}) \frac{\partial^2 \mathcal{L}}{\partial \Gamma^k_{ms} \partial \Gamma^i_{pl}} \right) y_l^j,$$

and

$$\Psi_{\lambda,\xi} = -\xi_j^m y_l^j x_m^i \frac{\partial \mathcal{L}}{\partial \Gamma^i_{kl}} \omega_k.$$

It is convenient to denote

$$\psi_m^j = y_l^j x_m^i \frac{\partial \mathcal{L}}{\partial \Gamma^i_{kl}} \omega_k.$$

Then $\Psi_{\lambda,\xi} = -\xi_j^m \psi_m^j$, and the first variation formula (11) can equivalently be written in the form $E_i^j(\mathcal{L})\xi_j^k x_k^i \omega_0 - \xi_j^k hd\psi_k^j = 0$. But the numbers $\xi_j^k \in \mathbb{R}$ are arbitrary, so we have

$$E_i^j(\mathcal{L}) x_k^i \omega_0 - hd\psi_k^j = 0. \tag{12}$$

Suppose now that a section γ satisfies the Euler-Lagrange equations. Then the form $E_i^j(\mathcal{L}) x_k^i \omega_0$ vanishes along $J^2\gamma$, so we have $J^2\gamma^* d\psi_k^j = dJ^2\gamma^*\psi_k^j = 0$. Integrating we can find an $(n-2)$-form η_k^j on U such that

$$J^1\gamma^* \psi_k^j = d\eta_k^j. \tag{13}$$

Conversely, if a section γ satisfies condition (13), then by (12), γ is necessarily an extremal.

This completes the proof. □

Remark 1. Condition (c) can be easily expressed in terms of components of the forms η_k^j.

Remark 2. Theorem 3 is also valid for second order lagrangians. The proof is based on the use of the Lepage form (4), and Lemma 6.

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