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Conformally Osserman manifolds and self-duality in Riemannian geometry

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Abstract. We study the spectral geometry of the conformal Jacobi operator on a 4-dimensional Riemannian manifold (M, g). We show that (M, g) is conformally Osserman if and only if (M, g) is self-dual or anti self-dual. Equivalently, this means that the curvature tensor of (M, g) is given by a quaternionic structure, at least pointwise.

Let $\mathcal{R}_g$ and $\mathcal{W}_g$ be the curvature operator and the Weyl curvature operator associated to a Riemannian manifold (M, g). The *Jacobi operator* $\mathcal{J}_{\mathcal{R}_g}$ and the *conformal Jacobi operator* $\mathcal{J}_{\mathcal{W}_g}$ are defined by:

$$\mathcal{J}_{\mathcal{R}_g}(x) : y \to \mathcal{R}_g(y, x)x \quad \text{and} \quad \mathcal{J}_{\mathcal{W}_g}(x) : y \to \mathcal{W}_g(y, x)x \,.$$

In contrast to the Jacobi operator, the *conformal Jacobi operator* is conformally invariant [**1**]; if $h = e^\alpha g$ is a conformally equivalent Riemannian metric, then

$$\mathcal{J}_{W_h} = \mathcal{J}_{W_g} \,.$$

One says that (M, g) is *Osserman* (resp. *conformally Osserman*) if the eigenvalues of $\mathcal{J}_{R_g}$ (resp. $\mathcal{J}_{W_g}$) are constant on the bundle of unit tangent directions. In a series of papers started by Chi [**3**] and continued by Nikolayevsky [**5, 6, 7**] it was shown that Osserman manifolds of dimension $n \neq 16$ are two-point homogeneous spaces; for $m \neq 16$, this gives an affirmative answer to a question raised by Osserman (see, for example, [**8**]).

Previous work [**1**] has shown that conformally Osserman manifolds are conformally flat if $m \equiv 1 \bmod 2$ and are either conformally flat or conformally equivalent to a complex space form (i.e. to complex projective space with the Fubini-Study metric or to the non-compact dual) if $m \equiv 2 \bmod 4$. In this present paper, we study the 4-dimensional setting motivated by work of Sekigawa and Vanhecke ([**9**]). Our main result is the following.

Theorem 1. *Let (M, g) be a 4-dimensional oriented Riemannian manifold. The following conditions are equivalent:*

(1) (M, g) is conformally Osserman.
(2) (M, g) is self-dual or anti-self dual.

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It is useful to work in a purely algebraic setting. Let V be a finite dimensional vector space equipped with a positive definite inner product $(\cdot,\cdot)$. We say that $R \in \otimes^4 V^*$ is an *algebraic* curvature tensor if R has the usual symmetries:

$$R(x,y,z,w) = -R(y,x,z,w) = R(z,w,x,y), \quad \text{and}$$
$$R(x,y,z,w) + R(y,z,x,w) + R(z,x,y,w) = 0\,.$$

The associated curvature operator $\mathcal{R}(x,y)$ is then characterized by the identity $(\mathcal{R}(x,y)z,w) = R(x,y,z,w)$. Let τ be the scalar curvature and let ρ be the Ricci operator. Set

$$\mathcal{R}_g^0(x,y)z := g(y,z)x - g(x,z)y, \quad \text{and}$$
$$\mathcal{L}(x,y)z := g(\rho y,z)x - g(\rho x,z)y + g(y,z)\rho x - g(x,z)\rho y\,.$$

The associated Weyl operator is given by

$$\mathcal{W}(x,y) := \mathcal{R}(x,y) + \frac{1}{(m-1)(m-2)}\tau\mathcal{R}_g^0(x,y) + \frac{1}{m-2}\mathcal{L}(x,y).$$

We say that R is *Osserman* (resp. *conformally Osserman*) if the eigenvalues of $\mathcal{J}_R$ (resp. $\mathcal{J}_W$) are constant on the unit sphere in V.

Let Φ be a skew-symmetric endomorphism of V with $\Phi^2 = -1$. Define

$$R_\Phi(x,y)z := (\Phi y,z)\Phi x - (\Phi x,z)\Phi y - 2g(\Phi x,y)\Phi z\,.$$

Then R_Φ is an *algebraic curvature tensor.* We say that $\{\Phi_1,\Phi_2,\Phi_3\}$ is a *unitary quaternion* structure on V if the Φ_i are self-adjoint and if the usual structure equations are satisfied:

$$\Phi_i\Phi_j + \Phi_j\Phi_i = -2\delta_{ij}\,\text{id}\,.$$

Theorem 1 is a consequence of the following purely algebraic fact:

THEOREM 2. *Let R be an algebraic curvature tensor on a 4-dimensional vector space V. The following assertions are equivalent:*

(1) R is conformally Osserman.
(2) R is self-dual or anti-self dual.
(3) There exists a unitary quaternion structure on V so that $W = \lambda_1 R_{\Phi_1} + \lambda_2 R_{\Phi_2} + \lambda_3 R_{\Phi_3}$ where $\lambda_1+\lambda_2+\lambda_3 = 0$.

PROOF. We begin by showing that the first assertion implies the second assertion in Theorem 2. Let R be a conformally Osserman algebraic curvature tensor on $\mathbb{R}^4$. Zero is always an eigenvalue of $\mathcal{J}_W$ since $\mathcal{J}_W(x)x = 0$. Let e_1 be a unit vector. Since $\mathcal{J}_W(\cdot)$ is symmetric, it has an orthonormal basis of real eigenvectors. Thus we may extend e_1 to an orthonormal basis $\{e_1,e_2,e_3,e_4\}$ so that

$$\mathcal{J}_W(e_1)e_2 = ae_2, \quad \mathcal{J}_W(e_1)e_3 = be_3, \quad \mathcal{J}_W(e_1)e_4 = ce_4\,.$$

Since $\text{Tr}(\mathcal{J}_W) = 0$, we have $a+b+c = 0$.

The argument given by Chi [**3**] in his analysis of the 4-dimensional setting was in part purely algebraic. This algebraic argument extends without change to this setting to show that, after possibly replacing $\{e_2,e_3,e_4\}$ by $\{-e_2,-e_3,-e_4\}$ that the non-vanishing components of the Weyl curvature on this basis are given by:

$$\begin{aligned} W_{1221} &= W_{3443} = -W_{1234} = a,\\ W_{1331} &= W_{2442} = -W_{1342} = b,\\ W_{1441} &= W_{2332} = -W_{1423} = c\,. \end{aligned} \tag{1}$$

Let $e^{ij} := e^i \wedge e^j$ where $\{e^i\}$ is the dual basis for V^*. We consider the following bases for $\Lambda_2^{\pm}(V)$:

$$f_1^{\pm} = e^{12} \pm e^{34},\ f_2^{\pm} = e^{13} \mp e^{24},\ f_3^{\pm} = e^{14} \pm e^{23}\,.$$

Since $\mathcal{W}(e^{pq}) = \frac{1}{2} W_{pqij} e^{ij}$,

$$\begin{array}{lll} \mathcal{W}(f_1^-) = 0, & \mathcal{W}(f_2^-) = 0, & \mathcal{W}(f_3^-) = 0, \\ \mathcal{W}(f_1^+) = -2af_1^+, & \mathcal{W}(f_2^+) = -2bf_2^+, & \mathcal{W}(f_3^+) = -2cf_3^+\,. \end{array}$$

Thus conformally Osserman algebraic curvature tensors are self-dual.

Next we show that the second assertion implies the third assertion in Theorem 2. Suppose that R is a self-dual algebraic curvature tensor on $\mathbb{R}^4$. Let e_1 be a unit vector. Choose an orthonormal basis $\{e_1, e_2, e_3, e_4\}$ for $\mathbb{R}^4$ so that

$$\mathcal{J}_W(e_1)e_2 = ae_2, \quad \mathcal{J}_W(e_1)e_3 = be_3, \quad \mathcal{J}W(e_1)e_4 = ce_4\,.$$

We then have

$$\begin{array}{lll} W_{1221} = a, & W_{1231} = 0, & W_{1241} = 0, \\ W_{1321} = 0, & W_{1331} = b, & W_{1341} = 0, \\ W_{1421} = 0, & W_{1431} = 0, & W_{1441} = c\,. \end{array} \tag{2}$$

By replacing $\{e_2, e_3, e_4\}$ by $\{-e_2, -e_3, -e_4\}$ if necessary, we can assume that $\{e_1, e_2, e_3, e_4\}$ is an oriented orthonormal basis. We have

$$\begin{aligned} \mathcal{W}(e^{12} - e^{34}) &= (W_{1212} - W_{3412})e^{12} + (W_{1234} - W_{3434})e^{34}, \\ &+ (W_{1213} - W_{3413})e^{13} + (W_{1214} - W_{3414})e^{14} \\ &+ (W_{1223} - W_{3423})e^{23} + (W_{1224} - W_{3424})e^{24}\,. \end{aligned}$$

Since W is self-dual, $\mathcal{W}(e^{12} - e^{34}) = 0$. Equation (2) then implies

$$\begin{array}{lll} W_{3412} = W_{1212} = -a, & W_{3434} = W_{1234} = -a, & W_{3413} = W_{1213} = 0, \\ W_{3414} = W_{1214} = 0, & W_{3423} = W_{1223} = 0, & W_{3424} = W_{1224} = 0\,. \end{array}$$

We argue similarly using $e^{13} + e^{24}$ and $e^{14} - e^{23}$ to see that the formulas of Equation (1) hold. We define a unitary quaternion structure by defining $\Phi_3 := \Phi_1\Phi_2$ where

$$\begin{array}{llll} \Phi_1 : e_1 \to e_2, & \Phi_1 : e_2 \to -e_1, & \Phi_1 : e_3 \to e_4, & \Phi_1 : e_4 \to -e_3, \\ \Phi_2 : e_1 \to e_3, & \Phi_2 : e_3 \to -e_1, & \Phi_2 : e_4 \to -e_2, & \Phi_2 : e_2 \to e_4\,. \end{array}$$

It is then immediate that the formulas of Equation (1) hold for $\tilde{W} := aW_{\Phi_1} + bW_{\Phi_2} + cW_{\Phi_3}$ and thus $W = \tilde{W}$. This shows that the second assertion implies the third assertion.

Finally, if W is given by a unitary quaternion structure, then the discussion of [**4**] shows that W is Osserman. This completes the proof of Theorem 2 and thereby of Theorem 1 as well. □

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