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Hidden symplecticity in Hamilton's principle algorithms

Roberto Benito and David Martín de Diego

Abstract. We treat the Lagrangian dynamics with higher-order constraints within a geometric framework which can be useful in the study of geometrical properties of variational integrators in the sense of [**7, 1**].

1. Introduction

Numerical methods for ordinary differential equations are of considerable practical importance. The main requirements are, of course, accuracy of the method, numerical stability, computational cost. But, also, there is the need to capture the qualitative behavior of the original system. This is the aim of geometric integration (see [**5**]) for recent developments). In particular in two recent articles [**7, 1**] appear a discussion of the use of Hamilton's principle to derive time-advance numerical algorithms for systems of ordinary differential equations which have a Lagrangian or Hamiltonian origin. In this methods, we first choose a class of functions (with undetermined parameters) for approximating the solution of the Hamiltonian or Lagrangian problem. Finally, Hamilton's principle is exactly applied to this space of approximating functions [**7**]. In [**1**] the methods based on Hamilton's principle are compared with traditional symplectic integrators, observing, in several examples, a better behavior of the first with respect to the second ones.

Now, in this paper, we analyze the geometric properties of numerical algorithms based on Hamilton's principle. The interesting result that we obtain is that there exists, for many cases of approximating functions, a "hidden symplecticity", in the sense, that it appears in the cotangent bundle of a higher-order tangent space ($T^*T^{(k-1)}Q$ where Q is the configuration manifold). For obtaining this new result in the numerical analysis literature is necessary to develop a geometrical formalism for higher-order variational calculus with constraints. To do that we adapt to the higher-order setting the recent construction developed by one of the authors and coworkers for vakonomic mechanics [**2**].

The paper is organized as follows. In Section 2 we review the construction and main properties of higher-order tangent bundles. Section 3 is devoted to introduce Hamilton's principle for higher-order Lagrangians with or without constraints. For problem with constraints, we use the classical treatment with Lagrange multipliers. In Section 4 we briefly present an intrinsic and geometric approach to higher-order constrained variational problems. Finally, in Section 5, we find the "symplectic character" of numerical algorithms based on Hamilton's principle for the case considered in [**7, 1**], polynomial functions. Of course, our methods may be used for different

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approximating spaces. Also, as a new line of research, we present optimal control of underactuated Lagrangian control systems as an interesting application of our developments.

2. Higher-order tangent bundles

Let Q be a manifold of dimension n. We introduce an equivalence relation in the set $C^\infty(\mathbb{R}, Q)$ of curves from $\mathbb{R}$ to Q. By definition, two given curves in Q, say, $\gamma_1(t)$ and $\gamma_2(t)$, $t \in (-a, a), a \in \mathbb{R}$ have a contact of order k at $q_0 = \gamma_1(0) = \gamma_2(0)$ if there is a local chart (φ, U) of Q such that $\gamma_1(0), \gamma_2(0) \in U$, and

$$\frac{d^s}{dt^s}\left(\varphi \circ \gamma_1(t)\right)_{|t=0} = \frac{d^s}{dt^s}\left(\varphi \circ \gamma_2(t)\right)_{|t=0},$$

for $s = 0, ..., k$. This is a well defined equivalence relation, and the equivalence class of a curve γ will be denoted by $[\gamma]_0^{(k)}$. The set of equivalence classes will be denoted by $T^{(k)}Q$ which is a differential manifold. Moreover, $\tau_Q^k : T^{(k)}Q \to Q$, where $\tau_Q^k\left([\gamma]_0^{(k)}\right) = \gamma(0)$, is a fiber bundle, called the *tangent bundle of order k* of Q. Define the surjective mappings: $\tau_Q^{(l,k)} : T^{(k)}Q \to T^{(l)}Q$, for $l \leqslant k$ given by $\tau_Q^{(l,k)}\left([\gamma]_0^{(k)}\right) = [\gamma]_0^{(l)}$. It is easy to see that $T^{(1)}Q \equiv TQ$, $T^{(0)}Q \equiv Q$, and $\tau_Q^{(0,k)} = \tau_Q^k$

From a function $f : Q \longrightarrow \mathbb{R}$ we construct the lifted functions

$$f^{(l,k)}([\gamma]_0^{(k)}) = \frac{d^l}{dt^l}\left(f \circ \gamma(t)\right)_{|t=0}, \qquad 0 \leqslant l \leqslant k.$$

Let X be a vector field on Q. We define its k-lift $X^{(k)}$ as the unique vector field satisfying

$$X^{(k)}(f^{(l,k)}) = (X(f))^{(l,k)}, \qquad \text{for all } f \in C^\infty(Q)$$

From a local chart (q^i) on Q we define induced coordinates $(q^{(0)i}, q^{(1)i}, \ldots, q^{(k)i})$ where $q^{(s)i} = (q^i)^{(s,k)}$, $1 \leqslant s \leqslant k$.

In coordinates, the k-lift of a vector field $X = X^i \frac{\partial}{\partial q^i}$ is

$$X^{(k)} = (X^i)^{(l,k)} \frac{\partial}{\partial q^{(l)i}}$$

Denote by $j_k : T^{(k)}Q \to T(T^{(k-1)}Q)$ the canonical immersion defined by

$$j_k([\gamma]_0^{(k)}) = [\gamma^{(k-1)}]_0^{(1)}$$

where $\gamma^{(k-1)}$ is the lift to $T^{(k-1)}Q$ of the curve γ, that is, the curve $\gamma^{(k-1)} : \mathbb{R} \to T^{(k-1)}Q$ is defined as $\gamma^{(k-1)}(t) = [\gamma_t]_0^{(k-1)}$ where $\gamma_t(s) = \gamma(t+s)$.

3. Hamilton's principle

Let $L : T^{(k)}Q \to \mathbb{R}$ be a lagrangian function of order k. Given two points $x, y \in T^{(k-1)}Q$ we define the manifold of $2k$-piecewise differentiable curves which connect x and y as

$$\mathcal{C}^{2k}(x,y) = \{c : [0,1] \longrightarrow Q \mid c \text{ is } C^{2k}([0,1], Q), c^{(k-1)}(0) = x, \text{ and } c^{(k-1)}(1) = y\}.$$

Fixed a curve c in $\mathcal{C}^{2k}(x,y)$, the tangent space to $\mathcal{C}^{2k}(x,y)$ at c is given by

$$\begin{aligned} T_c\mathcal{C}^{2k}(x,y) &= \Big\{X : [0,1] \longrightarrow TQ \mid X \text{ is } C^{2k-1}, X(t) \in T_{c(t)}Q, \\ &\qquad X^{(k-1)}(0) = 0, \text{ and } X^{(k-1)}(1) = 0\Big\}. \end{aligned}$$

Consider an **action functional** $\mathcal{J}$ on C^{2k}-curves in Q by integration of the Lagrangian L along the tangent to the curves:

$$\begin{array}{rccl} \mathcal{J}: & C^{2k}(x,y) & \longrightarrow & \mathbb{R} \\ & c & \longmapsto & \int_0^1 L(c^{(k)}(t))\, dt \end{array} \tag{1}$$

Hamilton's principle seeks the curves $c : [0,1] \to Q$ such that the functional $\mathcal{J}$ is stationary; that is, $d\mathcal{J}(c)(X) = 0$ for all $X \in T_c\mathcal{C}^{2k}(x,y)$. Taking a family of curves $c_\epsilon \in \mathcal{C}^{2k}(x,y)$, with $c_0 = c$, then the stationary condition is rewritten as

$$\frac{d}{d\epsilon}\Big|_{\epsilon=0}\mathcal{J}(c_\epsilon) = 0$$

Denote by $\delta c^i = \frac{d}{d\epsilon}\Big|_{\epsilon=0} c^i_\epsilon$ and by $\delta^{(l)}c^i = \frac{d^l}{dt^l}\delta c^i$, then using integration by parts several times we obtain

$$\frac{d}{d\epsilon}\Big|_{\epsilon=0}\mathcal{J}(c_\epsilon) = \int_0^1 \sum_{l=0}^{k}(-1)^l \frac{d^l}{dt^l}\left(\frac{\partial L}{\partial q^{(l)i}}\right)\cdot \delta c^i\, dt \tag{2}$$

$$+ \left[\sum_{l=0}^{k-1}\left[\sum_{s=0}^{k-l-1}(-1)^l \frac{d^s}{dt^s}\left(\frac{\partial L}{\partial q^{(l+s+1)i}}\right)\right]\cdot \delta^{(l)}c^i\right]_0^1 \tag{3}$$

But the last terms of (2) vanish since $\delta^{(l)}c^i(0) = \delta^{(l)}c^i(1) = 0$, $0 \leqslant l \leqslant k-1$.

Observe that the "boundary part" is the Poincaré-Cartan 1-form associated to L. Therefore, we have the following variational definition of the Euler-Lagrange-equations and Poincaré-Cartan 1-form:

DEFINITION 1. *Given a C^r lagrangian $L : T^{(k)}Q \to \mathbb{R}$, $r > k$, there exists a unique C^{r-k-1}- mapping*

$$\mathcal{E}L : T^{2k}Q \longrightarrow T^*Q$$

and a unique C^{r-k} 1-form Θ_L on $T^{2k-1}Q$ such that for all variations δc_ϵ we have

$$\frac{d}{d\epsilon}\Big|_{\epsilon=0}\mathcal{J}(c_\epsilon(t)) = \int_0^1 \mathcal{E}L(c^{(2k)}(t))\, dt + \Big[\Theta_L(c^{(2k-1)}(t)\cdot \delta^{(2k-1)}c(t)\Big]_0^1$$

Therefore it is possible to define the C^{l-k-1} 2-form $\Omega_L = -d\Theta_L$. If the Hessian matrix

$$\left(\frac{\partial^2 L}{\partial q^{(k)i}\partial q^{(k)j}}\right)$$

is regular then Ω_L is a symplectic form on $T^{(2k-1)}Q$.

In the following, assume that the Lagrangian L is regular. Take now the restriction $\mathcal{J}_L$ of the action functional $\mathcal{J}$ to the subspace $\mathcal{C}_L$ of solutions of the Euler-Lagrange equations. This space can be identified with the space of initial conditions $T^{(2k-1)}Q$ of the Euler-Lagrange equations. Therefore, it is easy to show that

$$d\mathcal{J}_L = F_t^*\Theta_L - \Theta_L$$

where F_t is the flow of the Euler-Lagrange vector field X_L, defined on $T^{(2k-1)}Q$ by $\mathcal{E}L \circ X_L = 0$. Since $d^2 = 0$ we deduce that the flow is symplectic, $F_t^* \Omega_L = \Omega_L$.

Moreover, if G is a Lie group of symmetries preserving the action functional and $\mathfrak{g}$ its Lie algebra then

$$0 = i_{\xi_Q^{(2k-1)}} d\mathcal{J}_L = i_{\xi_Q^{(2k-1)}} (F_t^* \Theta_L - \Theta_L) = F_t^*(i_{\xi_Q^{(2k-1)}} \Theta_L) - i_{\xi_Q^{(2k-1)}} \Theta_L$$

where ξ_Q is the infinitesimal generator associated to $\xi \in \mathfrak{g}$. Therefore, $J_\xi = i_{\xi_Q^{(2k-1)}} \Theta_L$ is a first integral of the flow. We conclude, after this exposition, that symplecticity and Noether's theorem are a consequence of the variational nature of the problem.

3.1. Higher-order variational problems with constraints. Now we place constraints considering M to be a submanifold of $T^{(k)}Q$ and restricting ourselves to curves lying on it. Assume that $(\tau_Q)_{|M} : M \longrightarrow Q$ is a surjective submersion. Consider the set (in the following assumed not empty):

$$\begin{aligned} \mathcal{C}^{2k}(x, y, M) \quad = \quad & \{c : [0,1] \longrightarrow Q \mid q \text{ is } C^{2k}, c^{(k-1)}(0) = x, \\ & \text{and } c^{(k-1)}(1) = y \text{ and } c^{(k)}(t) \in T^{(k-1)}Q \text{ for all } t \in [0,1]\}. \end{aligned}$$

DEFINITION 2. *A curve $c \in \mathcal{C}^{2k}(x, y, M)$ will be called a* **solution of the higher-order variational problem with constraints** *if $\mathcal{J}_{|\mathcal{C}^{2k}(x,y,M)}$ is stationary in c.*

The most useful result for this type of problems is the Lagrange Multipliers Lemma (see [**8**])

LEMMA 1. *Let N be a smooth manifold and let F be a Banach space with $g : N \to F$ a smooth submersion so that $g^{-1}(0)$ is a submanifold of N. Let $f : N \to \mathbb{R}$ be a smooth function. Then $n \in g^{-1}(0)$ is critical point of $f_{|g^{-1}(0)}$ if and only if there exists $\lambda \in F^*$ such that n is a critical point of $f - \lambda \circ g$.*

Consider that the constraint submanifold M is locally determined by the vanishing of m independent constraints Φ^α.

From, the Lagrange Multiplier Theorem we know that c is a (normal) critical curve for the higher-order variational problem with constraints if and only if c is a critical point of the functional

$$\mathcal{J}_M(c) = \int_0^1 L(q^{(k)}(t))\, dt - \lambda_\alpha g^\alpha(c)$$

where $\lambda_\alpha \in \mathcal{F}([0,1], \mathbb{R})^*$ and $g^\alpha : C^{2k}(x, y) \to \mathcal{F}([0,1], \mathbb{R})$ given by $g^\alpha(c) = \{t \longrightarrow \Phi^\alpha(c^{(k)}(t))\}$. Observe that $g : C^{2k}(x, y) \longrightarrow \mathcal{F}([0,1], \mathbb{R})^m$ is the smooth submersion which appears in the Lagrange Multiplier Theorem.

After a straightforward computation we deduce that c is a critical point of $\mathcal{J}_M$ if and only if

$$\begin{aligned} \frac{d}{d\epsilon}_{|\epsilon=0} \mathcal{J}_M(c_\epsilon(t)) \quad = \quad & \int_0^1 \sum_{l=0}^{k} (-1)^l \frac{d^l}{dt^l} \left(\frac{\partial L}{\partial q^{(l)i}} \right) - \frac{d^l}{dt^l} \left(\lambda_\alpha \frac{\partial \Phi^\alpha}{\partial q^{(l)i}} \right) \cdot \delta c^i \, dt \\ & + \left[\sum_{l=0}^{k-1} \left[\sum_{s=0}^{k-l-1} (-1)^l \frac{d^s}{dt^s} \left(\frac{\partial L}{\partial q^{(l+s+1)i}} - \lambda_\alpha \frac{\partial \Phi^\alpha}{\partial q^{(l+s+1)i}} \right) \right] \cdot \delta^{(l)} c^i \right]_0^1 \end{aligned}$$

for every variation c_ϵ.

As in the free case, new preservation properties arise.

4. Geometric formulation for higher order constrained variational problems

Now, we will develop a geometric characterization of vakonomic mechanics following an approach similar to the formulation given by Skinner and Rusk (see [**10**]).

Consider the Whitney sum $T^*(T^{(k-1)}Q) \oplus T^{(k)}Q$, and its canonical projections $pr_1 : T^*(T^{(k-1)}Q) \oplus T^{(k)}Q \longrightarrow T^*(T^{(k-1)}Q)$, $pr_2 : T^*(T^{(k-1)}Q) \oplus T^{(k)}Q \longrightarrow T^{(k)}Q$. Take now the submanifold $W_0 = pr_2^{-1}(M)$, where M is the constraint submanifold, locally determined by the constraint equations $\Phi^\alpha = 0$, $1 \leqslant \alpha \leqslant m$. Consider also the restrictions to W_0 of the canonical projections: $\pi_1 = pr_1\big|_{W_0}$ and $\pi_2 = pr_2\big|_{W_0}$. Now, define on $T^*(T^{(k-1)}Q) \times_{T^{(k-1)}Q} M$ the presymplectic 2-form $\Omega_{W_0} = \pi_1^*(\omega_{T^{(k-1)}Q})$, where $\omega_{T^{(k-1)}Q}$ is the canonical symplectic form on $T^*(T^{(k-1)}Q)$. Define also the function

$$H_{W_0} = \langle \pi_1, \pi_2 \rangle - \pi_2^* L|_M \ ,$$

where $\langle \cdot, \cdot \rangle$ denotes the natural pairing between vectors and covectors on $T^{(k-1)}Q$ (observe that elements of $T^{(k)}Q$ are considered as vectors on $T(T^{(k-1)}Q)$ by using j_k).

Now, we will see how the dynamics of the higher order constrained variational problem is determined by the solutions of the equation

$$i_X \Omega_{W_0} = dH_{W_0} \, . \tag{4}$$

Denote by $\Omega = pr_1{}^*(\omega_{T^{(k-1)}Q})$ (observe that the rank of this presymplectic form is equal to $2kn$ everywhere) and $H = \langle pr_1, pr_2 \rangle - pr_2^* L$. Then, it is easy to show that Equations (4) are equivalent to

$$\left\{ \begin{array}{rcl} i_X \Omega & = & dH + \lambda_\alpha d\Phi^\alpha \\ X & \in & TW_0 \end{array} \right. \tag{5}$$

Taking coordinates $(q^{(0)i}, q^{(1)i}, \ldots, q^{(k)i}, p_i^{(0)}, \ldots, p_i^{(k-1)})$ in $T^*(T^{(k-1)}Q) \oplus T^{(k)}Q$ then

$$\begin{array}{rcl} \Omega & = & \displaystyle\sum_{r=0}^{k-1} dq^{(r)i} \wedge dp_i^{(r)} \\ H & = & \displaystyle\sum_{r=0}^{k-1} q^{(r+1)i} p_i^{(r)} - L(q^{(0)i}, q^{(1)i}, \ldots, q^{(k)i}) \end{array}$$

Thus, if

$$X = \sum_{r=0}^{k} X^{(r)i} \frac{\partial}{\partial q^{(r)i}} + \sum_{r=0}^{k-1} Y_i^{(r)} \frac{\partial}{\partial p_i^{(r)}}$$

Then Equations (5) implies that

$$\begin{aligned} & X^{(r)i} = q^{(r+1)i}, \quad r = 0, \ldots, k-1 \\ & -Y_i^{(0)} = -\frac{\partial L}{\partial q^{(0)i}} + \lambda_\alpha \frac{\partial \Phi^\alpha}{\partial q^{(0)i}} \\ & -Y_i^{(r)} = p_i^{(r-1)} - \frac{\partial L}{\partial q^{(r)i}} + \lambda_\alpha \frac{\partial \Phi^\alpha}{\partial q^{(r)i}}, \qquad r = 1, \ldots, k-1 \end{aligned} \tag{6}$$

and the constraints:

$$
\begin{aligned}
&\Phi^\alpha = 0 \\
(7)\qquad &\varphi^1_i = p_i^{(k-1)} - \frac{\partial L}{\partial q^{(k)i}} + \lambda_\alpha \frac{\partial \Phi^\alpha}{\partial q^{(k)i}} = 0, \qquad i = 1, \ldots, n
\end{aligned}
$$

Therefore, the equations of motion of an integral curve of the solution X are:

$$
\begin{aligned}
&\dot{q}^{(r)i} = q^{(r+1)i}, \quad r = 0, \ldots, k-1 \\
(8)\qquad &-\dot{p}_i^{(0)} = -\frac{\partial L}{\partial q^{(0)i}} + \lambda_\alpha \frac{\partial \Phi^\alpha}{\partial q^{(0)i}} \\
&-\dot{p}_i^{(r)} = p_i^{(r-1)} - \frac{\partial L}{\partial q^{(r)i}} + \lambda_\alpha \frac{\partial \Phi^\alpha}{\partial q^{(r)i}}, \qquad r = 1, \ldots, k-1
\end{aligned}
$$

Differentiating (7) with respect to time, substituting in (8) and proceeding further, we find the equations of motion of the higher-order variational problem analized in Section 3.1, i.e.

$$
\sum_{r=0}^{k} (-1)^r \frac{d^r}{dt^r} \left(\frac{\partial L}{\partial q^{(r)i}} - \lambda_\alpha \frac{\partial \Phi^\alpha}{\partial q^{(r)i}} \right) = 0
$$

The new constraints $\varphi^1_i = 0$ determine the set of points W_1 of W_0 where (5) has a solution. This is precisely the first constraint submanifold for the presymplectic hamiltonian system $(W_0, \Omega_{W_0}, H_{W_0})$ (see, for instance, the Gotay-Nester constraint algorithm in **[3][4]**). The solution of Equation (4) on W_1 may not be tangent to W_1. In such a case, we have to restrict W_1 to the submanifold W_2 where this solution is tangent to W_1. Proceeding further, we obtain a sequence of submanifolds (assuming that all the subsets generated by the algorithm are submanifolds)

$$
\cdots \hookrightarrow W_k \hookrightarrow \cdots \hookrightarrow W_2 \hookrightarrow W_1 \hookrightarrow W_0 \ .
$$

Algebraically, these constraint submanifolds may be described as

(9)

$$
W_i = \{ x \in T^* \left(T^{(k-1)} Q \right) \times_{T^{(k-1)}Q} M \mid dH_{W_0}(x)(v) = 0 \ , \ \forall v \in T_x W_{i-1}^{\perp} \} \ , \quad i \geqslant 1 \ ,
$$

where $T_x W_{i-1}^{\perp} = \{ v \in T_x W_0 \mid \Omega_{W_0}(x)(u, v) = 0 \ , \ \forall u \in T_x W_{i-1} \}$. If the case when this constraint algorithm stabilizes, i.e., if there exists a positive integer $k \in \mathbb{N}$ such that $W_{k+1} = W_k$ and $\dim W_k \neq 0$, then we will have at least a well defined solution X on $W_f = W_k$ such that

$$
(i_X \Omega_{W_0} = dH_{W_0})_{|W_f} \ .
$$

As we will show in the next sections, a very important particular case is when the final constraint submanifold is W_1, that is, $W_f = W_1$. In the sequel we will assume that

$$
(10) \qquad \det \begin{pmatrix} \frac{\partial^2 L}{\partial q^{(k)i} \partial q^{(k)j}} & -\frac{\partial \Phi^\beta}{\partial q^{(k)i}} \\ \frac{\partial \Phi^\alpha}{\partial q^{(k)j}} & 0 \end{pmatrix} \neq 0
$$

If the matrix $\mathcal{W} = \left(\dfrac{\partial^2 L}{\partial q^{(k)i} \partial q^{(k)j}} \right)$ is non singular, we can re express the last condition on the determinant in the form $\det(\mathcal{C} \cdot W^{-1} \cdot \mathcal{C}^T)$ where $\mathcal{C}$ is the $m \times n$ matrix with entries $\mathcal{C}_{i\alpha} = (\dfrac{\partial \Phi^\alpha}{\partial q^{(k)i}})$. In that case, dimension of W_1 is even, $\dim W_1 = 2kn$. Observe that, applying the implicit function theorem to the constraint equations (7), we can locally express the Lagrange multipliers λ_α and the highest order velocities $q^{(k)i}$ in terms of the coordinates $(q^{(0)i}, q^{(1)i}, \ldots, q^{(k-1)i}, p_i^{(0)}, \ldots, p_i^{(k-1)})$, i.e.

$$\begin{aligned}\lambda_\alpha &= \lambda_\alpha(q^{(0)i}, q^{(1)i}, \ldots, q^{(k-1)i}, p_i^0, \ldots, p_i^{(k-1)})\\ q^{(k)i} &= q^{(k)i}(q^{(0)i}, q^{(1)i}, \ldots, q^{(k-1)i}, p_i^0, \ldots, p_i^{(k-1)})\end{aligned}$$

Therefore, we can also consider $(q^{(0)i}, q^{(1)i}, \ldots, q^{(k-1)i}, p_i^{(0)}, \ldots, p_i^{(k-1)})$ as local coordinates on W_1 . In such a case, the restriction Ω_{W_1} of the presymplectic form Ω_{W_0} is

$$\Omega_{W_1} = \sum_{r=0}^{k-1} dq^{(r)i} \wedge dp_i^{(r)}$$

From the above comments, is easy to deduce the following

PROPOSITION 1. *If, for any choice of coordinates*

$$(q^{(0)i}, q^{(1)i}, \ldots, q^{(k)i}, p_i^{(0)}, \ldots, p_i^{(k-1)})$$

in $T^*(T^{(k-1)}Q) \oplus T^{(k)}Q$ and every point in W_1,

$$det \begin{pmatrix} \frac{\partial^2 L}{\partial q^{(k)i}\partial q^{(k)j}} & -\frac{\partial \Phi^\beta}{\partial q^{(k)i}} \\ \frac{\partial \Phi^\alpha}{\partial q^{(k)j}} & 0 \end{pmatrix} \neq 0 \tag{11}$$

then (W_1, Ω_{W_1}) is a symplectic manifold.

5. Numerical algorithms based on Hamilton's principle

In Lewis and Kostelec [**7**] is given a general discussion of the use of Hamilton's variational principle to derive numerical methods for some systems of differential equations derived from a variational principle, in particular, Hamilton's equations and Euler-Lagrange equations. The comparison between these methods and symplectic algorithms was treated in [**1**]. For applying numerical methods based in Hamilton's principle, first, it is necessary to determine a class of functions with undetermined parameters for approximating the solution of the continuous equations over a fixed interval; finally, Hamilton's principle is applied exactly.

Consider a lagrangian function $L : T\mathbb{R}^n \longrightarrow \mathbb{R}$. The extremals of the functional

$$\mathcal{J}(c) = \int_{t_0}^{t_1} L(c(t), \dot{c}(t))\, dt$$

with initial and final given conditions $c(t_0) = q_0$ and $c(t_1) = q_1$ are the solutions to the Euler-Lagrange equations:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\right) - \frac{\partial L}{\partial q^i} = 0$$

The numerical approximation is derived by applying at each step Hamilton's principle over a particular subset of the C^2-functions. Habitually this subset of functions are polynomials taking fixed values at the initial point. For instance, using polynomials $P(t)$ of degree m satisfying the initial and final conditions, that is

$$P(t) = \frac{t_1 - t}{t_1 - t_0} q_0 + \frac{t - t_0}{t_1 - t_0} q_1 + (t_1 - t)(t - t_0) \sum_{j=0}^{m-2} \alpha_j t^j$$

The restriction to this particular set of polynomials of degree m is equivalent to impose the constraints

$$\Phi^i = q^{(m+1)i} = 0, \quad 1 \leqslant i \leqslant n$$

The equations of motion of an integral curve for the system determined by the lagrangian $L : TQ \longrightarrow \mathbb{R}$ and the constraints φ^i are

$$\begin{aligned} \dot{q}^{(r)i} &= q^{(r+1)i}, \quad r = 0, \dots, m \\ -\dot{p}_i^{(0)} &= -\frac{\partial L}{\partial q^i} \\ -\dot{p}_i^{(1)} &= p_i^{(0)} - \frac{\partial L}{\partial \dot{q}^i} \\ -\dot{p}_i^{(r)} &= p_i^{(r-1)}, \quad r = 2, \dots, m \end{aligned} \tag{12}$$

and the constraints

$$\begin{aligned} \Psi^i &= q^{(m+1)i} = 0, \quad 1 \leqslant i \leqslant n \\ \varphi_i^1 &= p_i^{(m)} + \lambda_i = 0, \quad 1 \leqslant i \leqslant n \end{aligned} \tag{13}$$

The matrix

$$\begin{pmatrix} \dfrac{\partial^2 L}{\partial q^{(k)i} \partial q^{(k)j}} & -\dfrac{\partial \Phi^\beta}{\partial q^{(k)i}} \\ \dfrac{\partial \Phi^\alpha}{\partial q^{(k)j}} & 0 \end{pmatrix} = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} \tag{14}$$

Therefore, the constraint algorithm stopped in the first constraint submanifold W_1 and moreover, applying Proposition 1, we deduce that Ω_{W_1} is symplectic.

6. Underactuated Lagrangian Control Systems

Consider, for simplicity, a configuration space $Q = Q_1 \times Q_2$, therefore $TQ = TQ_1 \times TQ_2$ where $T_{(q_1,q_2)}(Q_1 \times Q_2)$ is naturally identified with $T_{q_1}Q_1 \times T_{q_2}Q_2$. Assume that a lagrangian function $L : TQ \to \mathbb{R}$ is given and that the controlled external forces can be applied only to the first coordinates (q_1^i), while the coordinates (q_2^a) evolve freely. The equations of motions are:

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_1^i}\right) - \frac{\partial L}{\partial q_1^i} &= u^i, \\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_2^a}\right) - \frac{\partial L}{\partial q_2^a} &= 0 \end{aligned} \tag{15}$$

The optimization problem consists of finding a trajectory $(q_1^i(t), q_2^a(t), u^i(t))$ of state variables and control inputs satisfying Equations (15) from given initial data $(q_1(t_0), q_2(t_0), \dot{q}_1(t_0), \dot{q}_2(t_0))$ to final data $(q_1(t_f), q_2(t_f), \dot{q}_1(t_f), \dot{q}_2(t_f))$ and minimizing the cost functional

$$\mathcal{J} = \int_{t_0}^{t_f} C(q_1^i, q_2^a, \dot{q}_1^i, \dot{q}_2^a, u^i)\, dt$$

Since the control variables appears explicitly the previous optimal control problem is equivalent to the following constrained variational problem:

$$\widetilde{\mathcal{J}} = \int_{t_0}^{t_f} \widetilde{L}(q_1^i, q_2^a, \dot{q}_1^i, \dot{q}_2^a, \ddot{q}_1^i, \ddot{q}_2^a)\, dt$$

subject to the second order constraints:

$$\Phi^a(q_1^i, q_2^a, \dot{q}_1^i, \dot{q}_2^a, \ddot{q}_1^i, \ddot{q}_2^a) = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_2^a}\right) - \frac{\partial L}{\partial q_2^a} = 0$$

Here,

$$\widetilde{L}(q_1^i, q_2^a, \dot{q}_1^i, \dot{q}_2^a, \ddot{q}_1^i, \ddot{q}_2^a) = C(q_1^i, q_2^a, \dot{q}_1^i, \dot{q}_2^a, \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_1^i}\right) - \frac{\partial L}{\partial q_1^i}).$$

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Instituto de Matemáticas y Física Fundamental,
CSIC, Serrano 123,
28006 Madrid, Spain,

e-mail: `rbenito@imaff.cfmac.csic.es, d.martin@imaff.cfmac.csic.es`