DIFFERENTIAL GEOMETRY AND ITS APPLICATIONS
Proc. Conf. Prague, August 30 – September 3, 2004
Charles University, Prague (Czech Republic), 2005, 1 – 13

Almost Nilpotent Manifolds With Pinched Negative Curvature

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Abstract. The aim of this paper is to study geometry and topology of manifolds with pinched negative sectional curvature whose fundamental groups are almost nilpotent, especially such noncompact locally symmetric rank one manifolds. Such manifolds classify thin ends of geometrically finite manifolds with pinched negative sectional curvature. We describe them via fibre bundle structure. This study is based on our structural theorem for dynamics of discrete groups acting by isometries on nilpotent groups [**5**].

1. Introduction

For a given Riemannian manifold M with pinched negative sectional curvature, its universal cover is a pinched Hadamard manifold, i.e. a complete, simply connected Riemannian manifold X whose all sectional curvatures K lie between two negative constants, $-b^2 \leqslant K \leqslant -a^2$, where $0 < a < \infty$ and $0 < b < \infty$. Due to Margulis Lemma [**15, 6**], thin ends of such manifolds M can be described by parabolic subgroups of their fundamental groups discretely acting by isometries on X. Each such parabolic subgroup G is almost nilpotent and preserves setwise each of horospheres centered at a point $q \in \partial X$, the fixed point of G. In that way all thin ends of geometrically finite manifolds with pinched negative sectional curvature [**2, 7**] are classified by such manifolds with almost nilpotent fundamental groups. This and our geometric approach to such thin ends allowed us to prove that geometrically finite groups in such pinched Hadamard spaces are finitely presented [**5**] (this fact and the basics of geometry and topology of thin ends were known from our preprints since 1995-1997).

The main aim of this paper is to give topological and geometrical description (cf. [**11**]) of such almost nilpotent manifolds with pinched negative sectional curvature (in particular as fiber bundles) by using our structural theorem [**5**]:

Theorem 1. *Let $\mathcal{N}$ be a connected, simply connected nilpotent Lie group , C be a compact group of automorphisms of $\mathcal{N}$, and $G \subset \mathcal{N} \rtimes C$ be a discrete subgroup. Then there exist a connected Lie subgroup $\mathcal{N}_G$ of $\mathcal{N}$ and a finite index subgroup G^* of G with the following properties:*

- *There exists $b \in \mathcal{N}$ such that bGb^{-1} preserves $\mathcal{N}_G$;*

2000 *Mathematics Subject Classification.* 57, 55, 53, 51.

Key words and phrases. Manifolds of negative curvature, thin ends, discrete groups, nilpotent groups, Bieberbach theorems, fiber bundles.

This paper is in final form and no version of it will be published elswhere.

- *$\mathcal{N}_G/bGb^{-1}$ is compact;*
- *bG^*b^{-1} acts on $\mathcal{N}_G$ by left translations, and this action is free.*

This Bieberbach type theorem (cf. [**2**]) advances a result by Louis Auslander [**1**] who proved its claims (1) and (2) only for a finite index subgroup of a given discrete group G. Though our fundamental groups are almost nilpotent, we underline the importance of the Theorem 1 compactness condition on the group C of automorphisms of $\mathcal{N}$ by assuming the following condition **(N)** on our pinched Hadamard manifolds X:

- "**(N)**" The parabolic subgroup of $\mathrm{Isom}\, X$ that fixes a point $q \in \partial X$ at infinity acts transitively on each horosphere $X_t \subset X$ centered at q, which can be identified with a connected, simply connected nilpotent Lie group $\mathcal{N}$ with a compact group C of automorphisms.

Obviously, symmetric spaces of rank 1 with negative curvature are in our class of pinched Hadamard manifolds with N-property [**14**]. These are the hyperbolic spaces – either real, complex, quaternionic or octonionic ones, all their horospheres can be identified with connected simply connected Lie groups $\mathcal{N}$, and our discrete group G isometrically acts on $\mathcal{N}$ as a subgroup $G \subset \mathcal{N} \rtimes C$ where C is a compact group of automorphisms of $\mathcal{N}$. In the case of real hyperbolic spaces (of constant negative curvature), horospheres are flat, and G must be almost Abelian.

2. Geometry of pinched Hadamard spaces

Here we fix some notations and definitions, and briefly review some facts concerning geometry of pinched Hadamard spaces including negatively curved symmetric spaces of rank one and Carnot groups which correspond to horospheres in these spaces, see [**5**].

2.1. Horospherical coordinates on pinched Hadamard manifolds. Let X be a pinched Hadamard manifold with a distance function d and sectional curvature K, $-b^2 \leqslant K \leqslant -a^2$. Its infinity $X(\infty) = \partial X$ is homeomorphic to the sphere S^{n-1} where $n = \dim X$, and the isometry group $\mathrm{Isom}\, X$ acts at infinity as a convergence group [**10**]. So one can classify isometries $g \in \mathrm{Isom}\, X$ in the following three types. If g fixes a point in X, it is called *elliptic*. If g has exactly one fixed point, and it lies in ∂X, g is called *parabolic*. If g has exactly two fixed points, and they lie in ∂X, g is called *loxodromic*. These three types exhaust all the possibilities.

A subgroup $G \subset \mathrm{Isom}\, X$ is called *discrete* if it is a discrete subset of $\mathrm{Isom}\, X$. An infinite discrete group G is called *parabolic* if it has exactly one fixed point $\{p\} = \mathrm{fix}(G)$, $p \in \partial X$; then G consists of either parabolic or elliptic elements because a discrete group of isometries of a negatively curved space has no loxodromic and parabolic elements with a common fixed point, see [**9, 10**].

In order to study discrete parabolic subgroups in $\mathrm{Isom}\, X$, it is convenient to view X from a fixed point $q_\infty \in \partial X$. Then subspaces issuing from this point q_∞ can be used to define "horospherical coordinates" on X and the "upper half-space model" for X as follows [**5**].

Fixing a point $q \in \partial X$ at infinity of X, one can define two foliations of X. The first one consists of geodesics ending at q, $\ell : (-\infty, +\infty) \longrightarrow X$ where $\ell(t)$ converges to $q \in \partial X$ as t goes to ∞. The second foliation consists of horospheres centered at q. To define horospheres, we can use the Busemann function $F = \lim F_t$, where $F_t(x) = d(x, \ell(t)) - t$ with $\ell(t_0) = x \in X$, see [**9, 6**]. Then the horospheres centered at q are the level surfaces of the Busemann function F; they bound horoballs $F^{-1}(t_0,+\infty)$ centered at q. Due to **[13]**, the Busemann functions (and therefore horospheres) are at least C^2-smooth. Hence the notions of distance and geodesic curves are defined with respect to the induced metric which decreases exponentially as x approaches $q \in \partial X$, i.e. $t \to +\infty$, see **[13]** for estimates which depend on the pinching constants $a, b > 0$. As level surfaces of a Busemann function, horospheres are closed and therefore complete, and one always have minimal geodesics joining two points.

As we assumed in Introduction, our pinched Hadamard spaces X satisfy the condition **(N)**, i.e. each horosphere $X_t \subset X$ (identified with a connected, simply connected nilpotent Lie group $\mathcal{N}$) has a compact group C of automorphisms. Any automorphisms θ of such a nilpotent Lie group $\mathcal{N}$ induces an automorphism $\tilde{\theta}$ of the Lie algebra $\mathfrak{n}$ of the group $\mathcal{N}$. An automorphism θ is called semi-simple if the induced automorphism $\tilde{\theta}$ is semi-simple, that is, for each subspace L of the Lie algebra $\mathfrak{n}$ invariant under $\tilde{\theta}$ there is a complementary invariant subspace. It is well known **[17]** that an automorphism θ of a nilpotent Lie group $\mathcal{N}$ is semi-simple if it belongs to a compact group of automorphisms of $\mathcal{N}$. So there exists a left invariant metric on the group $\mathcal{N}$ such that $\mathcal{N} \rtimes C$ acts on $\mathcal{N}$ as a group of isometries. So any discrete subgroup of $\mathcal{N} \rtimes C$ can be viewed as a discrete isometry group of $\mathcal{N}$ with respect to that left invariant metric which can be viewed as a metric on each horosphere $X_t \subset X$, $X_t \cong \mathcal{N}$, preserved by the action of the parabolic group fixing q and isomorphic to $\mathcal{N} \rtimes C$. With respect to this metric, those various horospheres isometrically correspond via the geodesic perspective from q at infinity, $X_t \rangle X_s$. Furthermore, taking the composition $\exp \circ F \, : \, X \rangle (0,\infty)$ of the exponent and a Busemann function F at q, we may identify

$$\overline{X} \backslash \{q\} = X \cup \partial X \backslash \{q\} \rangle \mathcal{N} \times [0,\infty) \,, \tag{2.1}$$

and call this identification the *"upper half-space model"* for our pinched Hadamard manifold X that satisfies the condition **(N)**. The identification (2.1) and the standard coordinates on the nilpotent (Carnot) group $\mathcal{N}$ give us *horospherical coordinates* on X.

In particular, for negatively curved rank one symmetric spaces X, that is the hyperbolic spaces $H^n_{\mathbb{F}}$ associated with the real, complex, quaternions or Cayley numbers (octonions) $\mathbb{F}$, the sphere at infinity ∂X can be identified with one point compactification of the nilpotent group $\mathcal{N}$ in the Iwasawa decomposition of $\operatorname{Isom} X = \mathcal{KAN}$. In its turn, the nilpotent group $\mathcal{N}$ can be identified with the product $\mathbb{F}^{n-1} \times \operatorname{Im} \mathbb{F}$ equipped with the operations:

$$(\xi, v) \cdot (\xi', v') = (\xi + \xi', v + v' + 2 \operatorname{Im}\langle \xi, \xi' \rangle) \quad \text{and} \quad (\xi, v)^{-1} = (-\xi, -v) \,, \tag{2.2}$$

where $\langle , \rangle$ is the standard Hermitian product in $\mathbb{F}^{n-1}$, $\langle z, w \rangle = \sum z_i \overline{w_i}$. The group $\mathcal{N}$ is a 2-step nilpotent Carnot group with center $\{0\} \times \operatorname{Im} \mathbb{F} \subset \mathbb{F}^{n-1} \times \operatorname{Im} \mathbb{F}$, and acts on itself by the left translations $T_h(g) = h \cdot g \,, \quad h, g \in \mathcal{N}$.

In horospherical coordinates of a given rank one symmetric spaces $X = H^n_{\mathbb{F}}$, we identify $\overline{X} \backslash \{\infty\}$ with the product $\mathcal{N} \times [0,\infty)$, that is with $\mathbb{F}^{n-1} \times \operatorname{Im} \mathbb{F} \times [0,\infty)$. So in these coordinates, the left action (2.2) of the Carnot group $\mathcal{N}$ on itself extends to an isometric action (Carnot translation) on the $\mathbb{F}$-hyperbolic space in the following form:

$$T_{(\xi_0,v_0)}\,:\,(\xi,v,u)\longmapsto(\xi_0+\xi\,,v_0+v+2\,\mathrm{Im}\langle\xi_0,\xi\rangle\,,u)\,,\tag{2.3}$$

where $(\xi,v,u)\in\mathbb{F}^{n-1}\times\mathrm{Im}\,\mathbb{F}\times[0,\infty)$.

There are a natural norm and its induced distance on the Carnot group $\mathcal{N}=\mathbb{F}^{n-1}\times\mathrm{Im}\,\mathbb{F}$, which are known in the case of the Heisenberg group (when $\mathbb{F}=\mathbb{C}$) as the Cygan's norm and distance, see [**8, 3, 4**]. Namely, it assigns to $(\xi,v)\in\mathcal{N}$ the following non-negative real number:

$$|(\xi,v)|_c=(|\xi|^4+|v|^2)^{1/4}=|\,(|\xi|^2-v)|^{1/2}\,,\tag{2.4}$$

where $|.|$ is the norm in $\mathbb{F}$, see [**5**].

Using horospherical coordinates on $\overline{X}\backslash\{\infty\}=\mathbb{F}^{n-1}\times\mathrm{Im}\,\mathbb{F}\times[0,\infty)$, we can extend the norm (2.4) to a norm on the $\mathbb{F}$-hyperbolic space X:

$$|(\xi,v,u)|_c=|\,(|\xi|^2+u-v)|^{\frac12}\,.\tag{2.5}$$

This norm then gives rise to a distance on the rank one symmetric space $X=H^n_{\mathbb{F}}$ (on its upper half-space model $H^n_{\mathbb{F}}=\mathbb{F}^{n-1}\times\mathrm{Im}\,\mathbb{F}\times(0,\infty)$), which we still call the Cygan distance:

$$\begin{aligned}\rho_c\,:\,&\big(\mathbb{F}^{n-1}\times\mathrm{Im}\,\mathbb{F}\times(0,\infty)\big)\times\big(\mathbb{F}^{n-1}\times\mathrm{Im}\,\mathbb{F}\times(0,\infty)\big)\longrightarrow\mathbb{R}\\ &\rho_c\big((\xi,v,u),(\xi',v',u')\big)=\big|(T^{-1}_{(\xi',v')}(\xi,v),|u-u'|)\big|_c\\ &=\big|\,|\xi-\xi'|^2+|u-u'|-(v-v'+2\,\mathrm{Im}\langle\xi,\xi'\rangle)\big|^{\frac12}\,.\end{aligned}\tag{2.6}$$

In fact, it follows directly from the definition that Carnot translations and rotations are isometries of $H^n_{\mathbb{F}}=\mathbb{F}^{n-1}\times\mathrm{Im}\,\mathbb{F}\times(0,\infty)$ with respect to the Cygan distance ρ_c. Moreover, the restrictions of this distance to different horospheres centered at ∞ are the same, so Cygan distance plays the same role as Euclidean metric does on the upper half-space model for the real hyperbolic space $H^n_{\mathbb{R}}$.

Also we note that the relevant geometry on each horosphere $X_t\cong\mathcal{N}$ is the Carnot-Carathéodory geometry (cf. M.Gromov [**12**]) induced by the negatively curved metric of X. The geodesic perspective from q_∞ defines conformal maps between horospheres X_t and X_s which extends to conformal maps between the one-point compactifications $X_t\cup\infty$ homeomorphic to spheres S^{n-1}. In the limit, the induced metrics on horospheres fail to converge but the Carnot-Carathéodory structure remains fixed. In this way, the negatively curved geometry on X induces the Carnot-Carathéodory geometry on the sphere at infinity $\partial X\approx S^{n-1}$, naturally identified with the one-point compactification of the nilpotent (Carnot) group $\mathcal{N}$ (for symmetric spaces of rank 1, see Pansu [**16**]).

Considering the ball model $B^n_{\mathbb{F}}(0,1)\cong\mathbb{F}^{n-1}\times\mathrm{Im}\,\mathbb{F}\times(0,\infty)$ of a symmetric space X, we have a projection from $N\cup\infty=\partial(\mathbb{F}^{n-1}\times\mathrm{Im}\,\mathbb{F}\times(0,\infty))$ to the boundary of the unit ball $B^n_{\mathbb{F}}(0,1)\subset\mathbb{F}^n$, which is the extension to infinity of an isometry between two models of $\mathbb{H}^n_{\mathbb{F}}$ and has the form:

$$[z,t]\to\left[2\frac{1+|z|^2+t}{|1+|z|^2+t|^2}z\,,\ \frac{1+|z|^2+t}{|1+|z|^2+t|^2}(1-|z|^2+t)\right]\in S(0,1)\subset\mathbb{F}^n\,,\tag{2.7}$$

where $z\in\mathbb{F}^{n-1}$ and $t\in\mathrm{Im}\,\mathbb{F}$.

Note that the group K_0A fixing the origin of N and ∞ (or equivalently, fixing the geodesic in X ending at those two points at ∞) is $U(n-1)\times\mathbb{R}\times U(1)$ in $H^n_{\mathbb{C}}$, $Sp(n-1)\times\mathbb{R}\times Sp(1)$ in $H^n_{\mathbb{H}}$, and $Spin(7)\times\mathbb{R}$ in $H^2_{\mathbb{O}}$.

2.2. Geometry of thin ends of almost nilpotent manifolds. Here we shall recall how the structural Theorem 1 clarify the structure of thin ends of geometrically finite pinched negatively curved manifolds/orbifolds,or equivalently, the thin ends of almost nilpotent manifolds with pinched negative curvature and the N-property,in particular thin ends of locally symmetric spaces of rank one, see [**5**].

It is based on our geometric definition of parabolic cusp points (cusp ends) in pinched Hadamard manifolds X having N-property and is equivalent to another (not so transparent) dynamical approach [**7**]. Namely, suppose a point $p\in\partial X$ is a parabolic fixed point of a discrete group $\Gamma\subset\operatorname{Isom}X$ and let $G=\Gamma_p$ be the stabilizer of p in Γ (i.e., a maximal parabolic subgroup in Γ). Taking horospherical coordinates on X with respect to the point p at infinity (as in (2.1)), we can regard this stabilizer as $G\subset\mathcal{N}\rtimes C$ where C is a compact automorphism group of the connected Lie group $\mathcal{N}$ representing horospheres in X. Let $\rho_{\mathcal{N}}$ be a left invariant metric on the nilpotent Lie group $\mathcal{N}$, which is $\mathcal{N}\rtimes C$-invariant (due to compactness of C). Also, by $\mathcal{N}_G\subseteq\mathcal{N}$ we denote a minimal connected subgroup of the nilpotent group $\mathcal{N}\cong\partial X\backslash\{p\}$ preserved by the parabolic stabilizer G, and where the group G acts cocompactly, see Theorem 1.

Definition 1. *Given a positive number r and a parabolic fixed point $p\in\partial X$ of a discrete group $\Gamma\subset\operatorname{Isom}X$ with stabilizer $G=\Gamma_p\subset G$, the set*

$$U_{p,r}=\{x\in\partial X\backslash\{p\}\cong\mathcal{N}\,:\,\rho_{\mathcal{N}}(x,\mathcal{N}_G)\geqslant\frac{1}{r}\}\tag{2.8}$$

is called a standard cusp neighborhood of radius $r>0$ at p, provided it is precisely invariant with respect to the stabilizer G in Γ:

$$g(U_{p,r})=U_{p,r}\qquad \textit{for } g\in G=\Gamma_p\,,$$
$$\gamma(U_{p,r})\cap U_{p,r}=\emptyset\qquad \textit{for } \gamma\in\Gamma\backslash\Gamma_p\,.$$

As we have shown in [**5**], p is a parabolic cusp point of the discrete group Γ (cf. [**7**]) if and only if it has a standard cusp neighborhood $U_{p,r}$.

Now we can extend the $(\mathcal{N}\rtimes C)$-invariant metric $\rho_{\mathcal{N}}$ at infinity $\partial X\backslash\{p\}\cong\mathcal{N}$ to a $(\mathcal{N}\rtimes C)$-invariant metric ρ in $\overline{X}\backslash\{p\}=\mathcal{N}\times[0,\infty)$ as the $(\mathcal{N}\rtimes C)$-invariant product metric ρ in $\overline{X}\backslash\{p\}=\mathcal{N}\times[0,\infty)$ where $[0,\infty)$ has a metric commensurable with the square root of the Euclidean one (for symmetric spaces, see (2.6)).

Now we define standard cusp X-neighborhoods of radius $r>0$ as:

$$\hat{U}_{p,r}=\{x\in\overline{X}\backslash\{p\}\cong\mathcal{N}\times[0,\infty)\,:\,\rho(x,\mathcal{N}_G)\geqslant\frac{1}{r}\}\,,\tag{2.9}$$

whose boundaries at infinity coincide with the standard cusp neighborhoods $U_{p,r}\subset\partial X\backslash\{p\}$ in the Carnot group and which are precisely G_p-invariant, as $U_{p,r}$ are.

For a given discrete group $G\subset\mathcal{N}\rtimes C\subset\operatorname{Isom}X$ (for the almost nilpotent fundamental group of a given negatively curved manifold M), the quotient space with respect to its isometric action, $(X\cup\partial X\backslash\{\infty\})/G$, has a unique end. We call this end a *standard parabolic end* with a $(X,\operatorname{Isom}X)$-geometry. It is clear that neighborhoods of a standard parabolic end may be taken as $\hat{U}_{\infty,r}/G$, $r>0$.

We may represent a standard cusp X-neighborhood $\hat{U}_{p,r_0}$ at a cusp point p of a discrete group $\Gamma \subset \operatorname{Isom} X$ as the product

$$\hat{U}_{p,r_0} = S_{p,r_0} \times (0, r_0]\,, \tag{2.10}$$

if we foliate $\hat{U}_{p,r_0}$ by subsets $S_{p,r}$, $0 < r \leqslant r_0$, of the form:

$$S_{p,r} = \{x \in X \cup \partial X \backslash \{p\} \cong \mathcal{N} \times [0, \infty) \, : \, \rho(x, \mathcal{N}_{\Gamma_p}) = 1/r\}\,. \tag{2.11}$$

Since each set $S_{p,r}$ is Γ_p-invariant, we see that the standard cusp X-neighborhood $\hat{U}_{p,r_0}/\Gamma_p \subset M(\Gamma)$ of the cusp end E_p of the orbifold $M(\Gamma)$ is the product $(S_{p,r_0}/\Gamma_p) \times (0, 1]$. Furthermore, due to compactness of the automorphism group C of the nilpotent group $\mathcal{N}$, this foliation of a standard cusp X-neighborhood $\hat{U}_{p,r_0}$ by Γ_p-invariant sets $S_{p,r}$ defines a Γ_p-equivariant retraction

$$R_p \, : \, \hat{U}_{p,r_0} \rangle \mathcal{N}_{\Gamma_p}\,. \tag{2.12}$$

This retraction shows topological finiteness of ends of noncompact orbifolds $\mathcal{N}/G$ and X/G for discrete parabolic groups $G \subset \mathcal{N} \rtimes C \subset \operatorname{Isom} X$, which have the homotopy type of closed virtually nilpotent orbifolds $\mathcal{N}_G/G$. This shows that all discrete parabolic groups $G \subset \mathcal{N} \rtimes C$ (almost nilpotent fundamental groups of orbifolds with pinched negative curvature with the property N) are finitely presented.

3. Fibre bundle structures

Due to Margulis Lemma [**15, 6**], if $G \subset \operatorname{Isom} X$ is an almost nilpotent discrete group acting by isometries in a Hadamard space X with pinched negative sectional curvature, then either G preserves a geodesic in X or G fixes a unique point p at infinity of X. The first case when G is almost cyclic is well understood: X/G is a vector bundle over S^1. In the second case G is a parabolic subgroup preserving horospheres centered at p (and identified with a nilpotent Carnot group $\mathcal{N}$), i.e. $G \subset \mathcal{N} \rtimes C \subset \operatorname{Isom} X$, and as we have shown in the previous section, any manifold/orbifold X/G with pinched negative curvature (and the property N) is the product $\mathcal{N}/G \times \mathbb{R}$ of the quotient of the Carnot group $\mathcal{N}/G$ and the real line. Let us call those quotients as locally Carnot spaces. Our goal here is to show (by applying Theorem 1 and extending our arguments in [**5**]) that, up to a finite covering, such a locally Carnot space $\mathcal{N}/G$ is a trivial vector bundle over a compact infra-nilmanifold. In the case of almost nilpotent locally symmetric rank one spaces, we will show that such a compact infra-nilmanifold is the total space of a torus bundle over a torus. Here dimensions of the fibers and the base torus of these torus bundles depend on the Carnot group $\mathcal{N} = \mathbb{F}^{n-1} \times \operatorname{Im} \mathbb{F}$ and the fundamental group $G \subset \mathcal{N} \rtimes C$ where the compact group C of automorphisms of $\mathcal{N}$ is either $\mathrm{U}(n-1)$ if $\mathbb{F} = \mathbb{C}$, or $\mathsf{p}(n-1)$ if $\mathbb{F} = \mathbb{H}$, or $\mathrm{Spin}(7)$ if $\mathbb{F} = \mathbb{O}$.

We start with some preliminary statements.

LEMMA 1. *Let $G \subset \mathcal{N} \rtimes C$ be a torsion-free discrete group acting on the Carnot group $\mathcal{N} = \mathbb{F}^{n-1} \times \operatorname{Im} \mathbb{F}$ with non-compact quotient. Then the Carnot manifold $\mathcal{N}/G$ is a vector bundle over a compact manifold.*

PROOF. According to Theorem 1, G preserves a connected Lie subgroup $\mathcal{N}_G \subset \mathcal{N}$ with compact quotient $\mathcal{N}_G/G$. There is an alternative for $\mathcal{N}_G$: either $\{0\} \times \operatorname{Im} \mathbb{F} \subset \mathcal{N}_G$ or $\{0\} \times \operatorname{Im} \mathbb{F} \not\subset \mathcal{N}_G$.

Assuming the first case, that is $\{0\} \times \operatorname{Im}\mathbb{F} \subset \mathcal{N}_G$, we consider the natural projection $P : \mathbb{F}^{n-1} \times \operatorname{Im}\mathbb{F} \longrightarrow \mathbb{F}^{n-1}$ and the orthogonal complement $\mathcal{N}_G^{\perp}$ to $P(\mathcal{N}_G)$ in $\mathbb{F}^{n-1}$. Let us define another projection π,

$$\pi : \mathbb{F}^{n-1} \times \operatorname{Im}\mathbb{F} = (P(\mathcal{N}_G) \oplus \mathcal{N}_G^{\perp}) \times \operatorname{Im}\mathbb{F} \longrightarrow \mathcal{N}_G \,,$$

$$\pi(\xi + \xi',\ v) = (\xi,\ v + 2\operatorname{Im}\langle \xi', \xi\rangle) \tag{3.1}$$

where $\xi \in P(\mathcal{N}_G)$ and $\xi' \in \mathcal{N}_G^{\perp}$.

Then, for any $(\xi_0, v_0) \in \mathcal{N}_G$, the preimage

$$\pi^{-1}(\xi_0, v_0) = \{(\xi_0 + \xi',\ v_0 - 2\operatorname{Im}\langle \xi', \xi_0\rangle) \ :\ \xi' \in \mathcal{N}_G^{\perp}\} \tag{3.2}$$

is an affine subspace passing through (ξ_0, v_0), and the map from $\mathcal{N}_G^{\perp}$ to $\pi^{-1}(\xi_0, v_0)$,

$$\xi' \longmapsto (\xi_0 + \xi',\ v_0 - 2\operatorname{Im}\langle \xi', \xi_0\rangle)\,,$$

converts $\pi : \mathbb{F}^{n-1} \times \operatorname{Im}\mathbb{F} \to \mathcal{N}_G$ into a vector bundle. It is easy to check that π is G-equivariant, and the G-action on $\mathbb{F}^{n-1} \times \operatorname{Im}\mathbb{F}$ preserves the vector space structure on the fibers of π. Thus π induces a map (also denoted as π):

$$\pi : \mathcal{N}/G \longrightarrow \mathcal{N}_G/G\,,$$

which is a vector bundle.

Notice that the G-action preserves the metric on the fibers $\pi^{-1}(\xi_0, v_0)$, $(\xi_0, v_0) \in \mathcal{N}_G$, induced by the metric on $\mathcal{N}_G^{\perp}$. So we have a natural metric on the vector bundle $\pi : \mathcal{N}/G \to \mathcal{N}_G/G$.

Now let us suppose that the second case occurs, $\{0\} \times \operatorname{Im}\mathbb{F} \not\subset \mathcal{N}_G$.

In this case, the group $\mathcal{N}_G$ must be Abelian, so $P(\mathcal{N}_G)$ is totally real in $\mathbb{F}^{n-1}$, that is $\langle v, w\rangle$ is real for any $v, w \in P(\mathcal{N}_G)$. Therefore, for some fixed $a \in \mathbb{R}^k$, we can represent our subspaces in the form:

$$P(\mathcal{N}_G) = \mathbb{R}^k \times \{0\} \subset \mathbb{R}^{n-1} \subset \mathbb{F}^{n-1} \quad \text{and} \quad \mathcal{N}_G = \{(x, \langle x, a\rangle) \ :\ x \in \mathbb{R}^k\}\,.$$

We then define an analogue of the projection (3.1), $\pi : \mathbb{F}^{n-1} \times \operatorname{Im}\mathbb{F} \longrightarrow \mathcal{N}_G$, as

$$\pi : \mathbb{F}^{n-1} \times \operatorname{Im}\mathbb{F} = (P(\mathcal{N}_G) \oplus \mathcal{N}_G^{\perp}) \times \operatorname{Im}\mathbb{F} \longrightarrow \mathcal{N}_G\,, \quad \pi(\xi + \xi', v) = (\xi, \langle \xi, a\rangle) \tag{3.3}$$

where $\xi \in P(\mathcal{N}_G), \xi' \in \mathcal{N}_G^{\perp}$.

As before in (3.2), for any $(x_0, \langle x_0, a\rangle) \in \mathcal{N}_G$, the preimage

$$\pi^{-1}(x_0, \langle x_0, a\rangle) = \{(x_0 + \xi', v) \ :\ \xi' \in \mathcal{N}_G^{\perp}, v \in \operatorname{Im}\mathbb{F}\} \tag{3.4}$$

is an affine subspace passing through the point $(x_0, \langle x_0, a\rangle)$, and the map from $\mathcal{N}_G^{\perp} \times \operatorname{Im}\mathbb{F}$ to $\pi^{-1}(x_0, \langle x_0, a\rangle)$,

$$(\xi', v) \longmapsto (x_0 + \xi',\ \langle x_0, a\rangle + v)$$

converts $\pi : \mathbb{F}^{n-1} \times \operatorname{Im}\mathbb{F} \to \mathcal{N}_G$ into a vector bundle. It is easy to check that this projection π is G-equivariant, and the G-action preserves the vector space structure on the fibers of π. Thus π induces a map (still denoted as π):

$$\pi : \mathcal{N}/G \longrightarrow \mathcal{N}_G/G$$

which is a vector bundle, too.

So in both cases, $\mathcal{N}/G$ has a natural vector bundle structure over the compact manifold $\mathcal{N}_G/G$. $\square$

It follows from this Lemma that any Carnot manifold is homotopy equivalent to a compact manifold. It implies

Corollary 1. *The fundamental group of a Carnot manifold is finitely presented.*

According to Theorem 1, the compact quotient $\mathcal{N}_G/G$ is finitely covered by a nilmanifold $\mathcal{N}_G/G^*$ where G^* acts on $\mathcal{N}_G$ by translations. The next lemma describes the structure of the covering nilmanifold $\mathcal{N}_G/G^*$.

Lemma 2. *Let V be a connected Lie subgroup of the Carnot group $\mathcal{N} = \mathbb{F}^{n-1} \times \operatorname{Im}\mathbb{F}$ and $G \subset V$ a discrete co-compact subgroup of V. Then the manifold V/G is*

- *a torus if V is Abelian;*
- *the total space of a torus bundle over a torus if V is not Abelian.*

Proof. The first assertion for the Abelian case is clear because G acts on V as a group of Euclidean translations.

In the second case of non-Abelian V, the group G is not Abelian because it is a lattice in V. So G is a discrete torsion-free 2-step nilpotent group, and its center $C(G)$ and $G/C(G)$ are both free Abelian groups.

As V is not Abelian, $\{0\} \times \mathbb{R} \subset V$. Similarly, as the group G is not Abelian, G contains elements of the form $(0, v)$, $v \neq 0$, which belong to the center $C(G)$. Let us consider V as a vector subspace in $\mathcal{N}$ and V_C be its subspace spanned by $C(G)$. We have that $\{0\} \times \mathbb{R} \subset V_C$. For the natural projection $P : \mathbb{F}^{n-1} \times \operatorname{Im}\mathbb{F} \longrightarrow \mathbb{F}^{n-1}$, let $V_C^{\perp}$ be the orthogonal complement of $P(V_C)$ in $P(V)$ and π the following projection:

$$\pi : V \longrightarrow V_C^{\perp}\,, \quad \pi(\xi + \xi^{\perp}, v) = \xi^{\perp} \tag{3.5}$$

where $\xi \in P(V_C), \xi^{\perp} \in V_C^{\perp}$.

Notice that π projects the group G onto a subgroup $\pi(G)$ of $V_C^{\perp}$, which is considered as an Abelian group under vector addition (not as a subgroup of $\mathcal{N}$). In fact, this subgroup $\pi(G) \subset V_C^{\perp}$ is discrete. Otherwise, there would exist a sequence of distinct elements $g_i = (\xi_i + \xi_i^{\perp}, v_i) \in G$ such that $\xi_i^{\perp} \neq 0$ for all i and $\lim \xi_i^{\perp} = 0$. On the other hand, as $C(G)$ spans V_C, $V_C/C(G)$ is compact. So there exists a sequence of elements $h_i = (\xi_i', v_i') \in C(G)$ such that the set $\{h_i(\xi_i, v_i)\}$ lies in a compact set. Therefore, the set of elements

$$h_i \cdot g_i = (\xi_i', v_i') \cdot (\xi_i + \xi_i^{\perp}, v_i) = (\xi_i' + \xi_i + \xi_i^{\perp}, v_i' + v_i)$$

with distinct $\xi^{\perp}$-components lies in a compact set. Thus $\{h_i \cdot g_i\}$ is a bounded sequence of distinct elements of the discrete group G. This contradiction with discreteness of G shows that $\pi(G) \subset V_C^{\perp}$ is discrete.

Furthermore, for any $g \in G$, the following diagram is commutative:

$$\begin{tikzcd} V \arrow[r, "\pi"] \arrow[d, "g"'] & V_C^{\perp} \arrow[d, "\pi(g)"] \\ V \arrow[r, "\pi"'] & V_C^{\perp} \end{tikzcd}$$

Thus the projection π is G-equivariant and is projected to a continuous map

$$\bar{\pi} : V/G \longrightarrow V_C^{\perp}/\pi(G)\,.$$

Since V/G is compact, the quotient $V_C^{\perp}/\pi(G)$ is compact, and $\pi(G)$ is an Abelian lattice in the Euclidean space $V_C^{\perp}$. So, the quotient space $V_C^{\perp}/\pi(G)$ is a torus.

The fiber of $\bar{\pi}$ is $\bar{\pi}^{-1}(0) = V_C/C(G)$ and is in fact a torus as V_C is Abelian. $\square$

THEOREM 2. *Let $G \subset \mathcal{N} \rtimes C$ be a torsion-free discrete group acting on the Carnot group $\mathcal{N} = \mathbb{F}^{n-1} \times \operatorname{Im} \mathbb{F}$ with non-compact quotient. Then the manifold $\mathcal{N}/G$ has zero Euler characteristic and is a vector bundle over a compact infra-nilmanifold. Furthermore, this compact infra-nilmanifold is finitely covered by a nilmanifold which is either a torus or the total space of a torus bundle over a torus.*

PROOF. Due to Lemma 2, it is enough to show that the Carnot manifold $\mathcal{N}/G$ has zero Euler characteristic.

As before, let $G^* \subset G$ be a finite index subgroup acting co-compactly on the G-invariant subspace $\mathcal{N}_G \subset \mathcal{N}$ provided by Theorem 1. Due to Lemma 3, $\mathcal{N}/G$ is homotopy equivalent to $\mathcal{N}_G/G$ which is finitely covered by $\mathcal{N}_G/G^*$. Since the total space of any torus bundle has zero Euler characteristic, we have $\chi(\mathcal{N}_G/G^*) = 0$ and hence $\chi(\mathcal{N}/G) = 0$. $\square$

We remark that, in general, the vector bundle $\mathcal{N}/G \to \mathcal{N}_G/G$ is non-trivial. The simplest example of such non-trivial vector bundles is given for the Heisenberg group $\mathcal{N} = \mathcal{H}_{2n-1}$, see Example 4.6 in [**4**]. One can easily generalize it for other Carnot groups $\mathcal{N} = \mathbb{F}^{n-1} \times \operatorname{Im} \mathbb{F}$. However, up to a finite covering, the bundle $\mathcal{N}/G \to \mathcal{N}_G/G$ is trivial. To show this, we need to recall the notion of fiber products.

Let $H \subset F$ be a closed Lie subgroup of a Lie group F and V a finite dimensional real vector space. Let us also suppose that we have a Lie group homomorphism $\rho : H \to GL(V)$. Then one can define an H-action on the product $F \times V$ by the rule $h \cdot (f, v) = (hf, \rho(h)v)$. The corresponding quotient space $(F \times V)/H$ (with the quotient topology) is called a *fiber product* and is denoted as $F \times_H V$.

Let $[f, v]$ denote the equivalence class of a pair (f, v). Then, due to the definition, two equivalence classes coincide, $[f, v] = [k, u]$, if and only if there exists $h \in H$ such that $f = hk$ and $v = \rho(h)u$. For the homogeneous space F/H consisting of right cosets of H, we have a map

$$\Pi : F \times_H V \to F/H, \quad \Pi[f, v] = Hf$$

whose fibers $\Pi^{-1}[f]$ may be (non-canonically) identified with V. This converts $F \times_H V$ into a vector bundle over F/H with rank equal to $\dim V$.

LEMMA 3. *Let $F \times_H V$ be a fiber product and suppose that the homomorphism $\rho : H \to GL(V)$ extends to a homomorphism $\rho : F \to GL(V)$. Then $F \times_H V$ is a trivial bundle, $F \times_H V \cong F/H \times V$.*

PROOF. The isomorphism $F \times_H V \cong F/H \times V$ is given by $[f, v] \to (Hf, \rho(f)^{-1}(v))$. $\square$

THEOREM 3. *Let $G \subset \mathcal{N} \rtimes C$ be a discrete group acting on the Carnot group $\mathcal{N} = \mathbb{F}^{n-1} \times \operatorname{Im} \mathbb{F}$ with non-compact quotient, and $\mathcal{N}_G \subset \mathcal{N}$ a connected G-invariant Lie subgroup on which G acts co-compactly. Then there exists a finite index subgroup $G_0 \subset G$ such that the vector bundle $\mathcal{N}/G_0 \to \mathcal{N}_G/G_0$ is trivial. In particular, any Carnot orbifold $\mathcal{N}/G$ is finitely covered by the product of a compact nilmanifold $\mathcal{N}_G/G_0$ and an Euclidean space.*

PROOF. Passing to a finite index subgroup, we may assume that the group G is torsion-free. Then we shall find a finite index subgroup $G_0 \subset G$ whose rotational part is "good". After that we shall express the vector bundle $\mathcal{N}/G_0 \to \mathcal{N}_G/G_0$ as the Whitney sum of a trivial bundle and a fiber product. To finish the proof, we will use Lemma 3 to show the triviality of fiber products.

To realize this scheme we consider a finite index subgroup $G^* \subset G$ (provided by Theorem 1) and a G-invariant connected Lie subgroup $\mathcal{N}_G \subset \mathcal{N}$ where G^* acts co-compactly by translations. From the proof of Theorem 1 (see [**5**]), it follows that the rotational part of G^* is contained in a toral group, and there is the following alternative for $\mathcal{N}_G$: either $\{0\} \times \mathbb{R} \subset \mathcal{N}_G$ or $\{0\} \times \mathbb{R} \not\subseteq \mathcal{N}_G$.

Suppose we have the first case, $\{0\} \times \mathbb{R} \subset \mathcal{N}_G$. Let $P : \mathbb{F}^{n-1} \times \operatorname{Im}\mathbb{F} \longrightarrow \mathbb{F}^{n-1}$ be the natural projection, $P(\xi, v) = \xi$. Then $\mathcal{N}_G = P(\mathcal{H}_G) \times \mathbb{R}^k$ where $\{0\} \times \mathbb{R}^k \subset \{0\} \times \operatorname{Im}\mathbb{F}$ and $k \geqslant 1$.

Let $\mathcal{N}_\mathbb{F}$ be the $\mathbb{F}$-linear subspace of $\mathbb{F}^{n-1}$ spanned by $P(\mathcal{N}_G)$, and $\mathcal{N}_\mathbb{F}^\perp$ the orthogonal complement of $\mathcal{N}_\mathbb{F}$ in $\mathbb{F}^{n-1}$ with respect to the standard Hermitian product. For any $g = (\xi, v) \cdot A \in G^*$, restriction of its rotational part A to $P(\mathcal{N}_G)$ is the identity because G^* acts on $\mathcal{N}_G$ by translations. Since $A \in C$ is $\mathbb{F}$-linear, $A|_{\mathcal{N}_\mathbb{F}} = \mathrm{id}$.

Let H_1 be the orthogonal complement of $P(\mathcal{N}_G)$ in $\mathcal{N}_\mathbb{F}$, and $\{e_1, \ldots, e_q\}$ a basis of H_1. For each $i = 1, \ldots, q$, we define a map (compare (3.2)):

$$s_i : \mathcal{N}_G \longrightarrow \mathbb{F}^{n-1} \times \operatorname{Im}\mathbb{F}, \quad s_i(\xi, v) = (\xi + e_i\,,\, v - 2\operatorname{Im}\langle e_i, \xi\rangle)\,.$$

It is easy to check that s_i is G^*-equivariant, so s_i induces a section

$$\sigma_i : \mathcal{N}_G/G^* \longrightarrow \mathcal{N}/G^*$$

of the vector bundle $\pi : \mathcal{N}/G^* \to \mathcal{N}_G/G^*$. Since $\{s_1, \ldots, s_q\}$ are linearly independent, $\{\sigma_1, \ldots, \sigma_q\}$ spans a trivial subbundle F of $\pi : \mathcal{N}/G^* \to \mathcal{N}_G/G^*$.

As in the proof of Lemma 3, we may consider a metric on the vector bundle $\pi : \mathcal{N}/G^* \to \mathcal{N}_G/G^*$, which is induced by the metric on $\mathcal{N}_G^\perp = \mathcal{N}_\mathbb{F}^\perp \oplus H_1$. Let $F^\perp$ be the orthogonal complement of F with respect to this metric. Then

$$F^\perp = \{(\xi, v) \in \mathbb{F}^{n-1} \times \operatorname{Im}\mathbb{F} : \xi \in P(\mathcal{N}_G) \oplus \mathcal{N}_\mathbb{F}^\perp\}/G^*\,, \tag{3.6}$$

and $\mathcal{N}/G^* = F \oplus F^\perp$ where the bundle F is trivial.

Now let us show that $F^\perp$ is a fiber product. Since G^* is torsion free, the map

$$i : G^* \longrightarrow \mathcal{N}_G\,, \qquad i(g) = (\xi, v)\,, \quad \text{for } g = (\xi, v) \cdot A \in G^*$$

is an embedding. Here, in the representation $g = (\xi, v) \cdot A \in G^*$, we have that $(\xi, v) \in \mathcal{N}_G$ and $A|_{P(\mathcal{N}_G)} = \mathrm{id}$. Hence $A|_{\mathcal{N}_\mathbb{F}} = \mathrm{id}$, and A preserves $\mathcal{N}_\mathbb{F}^\perp$.

Let us define a homomorphism

$$\rho : i(G^*) \longrightarrow GL(\mathcal{N}_\mathbb{F}^\perp)\,, \quad \rho(i(g)) = A|_{\mathcal{N}_\mathbb{F}^\perp} \quad \text{for } g = (\xi, v) \cdot A \in G^*\,. \tag{3.7}$$

This homomorphism defines a fiber product $\mathcal{N}_G \times_{i(G^*)} \mathcal{N}_\mathbb{F}^\perp$ where the action of $i(G^*)$ on $\mathcal{N}_G \times \mathcal{N}_\mathbb{F}^\perp$ is as follows:

$$\begin{aligned} i(g)((\xi', v'),\ \xi^\perp) &= (i(g) \cdot (\xi', v'),\ \ \rho(i(g))\xi^\perp) = ((\xi, v) \cdot (\xi', v'),\ \ A\xi^\perp) \\ &= (\ (\xi + \xi', v + v' + 2\operatorname{Im}\langle \xi, \xi'\rangle)\,,\ \ A\xi^\perp) \end{aligned} \tag{3.8}$$

where $g = (\xi, v) \cdot A \in G^*$, $(\xi', v') \in \mathcal{N}_G$ and $\xi^\perp \in \mathcal{N}_\mathbb{F}^\perp$.

Using the mapping $((\xi', v'),\ \xi^\perp) \to (\xi' + \xi^\perp, v')$, we can identify $\mathcal{N}_G \times \mathcal{N}_\mathbb{F}^\perp$ with

$$\{(\xi, v) \in \mathbb{F}^{n-1} \times \operatorname{Im}\mathbb{F} :\ \xi \in P(\mathcal{N}_G) \oplus \mathcal{N}_\mathbb{F}^\perp\}\,.$$

Then the $i(G^*)$-action in (3.8) becomes

$$i(g)(\xi' + \xi^\perp, v') = (\xi + \xi' + A\xi^\perp\,,\ v + v' + 2\operatorname{Im}\langle \xi, \xi'\rangle)$$

$$= (\xi + \xi' + A\xi^{\perp}\,,\ v + v' + 2\,\mathrm{Im}\langle \xi, \xi' + A\xi^{\perp}\rangle)\,, \tag{3.9}$$

where $\langle \xi, A\xi^{\perp}\rangle = 0$ because $\mathcal{N}_{\mathbb{F}}^{\perp}$ is orthogonal to $\mathcal{N}_{\mathbb{F}}$. Moreover, this action is the same as the action of g on $\{(\xi, v) \in \mathbb{F}^{n-1} \times \mathrm{Im}\,\mathbb{F}\,:\ \xi \in P(\mathcal{N}_G) \oplus \mathcal{N}_{\mathbb{F}}^{\perp}\,,\ v \in \mathrm{Im}\,\mathbb{F}\}$:

$$\begin{aligned} g(\xi' + \xi^{\perp}, v') &= (\xi, v)\cdot A(\xi' + \xi^{\perp}, v') \\ &= (\xi, v)(A\xi' + A\xi^{\perp}, v') = (\xi, v)(\xi' + A\xi^{\perp}, v') \\ &= (\xi + \xi' + A\xi^{\perp},\ v + v' + 2\,\mathrm{Im}\langle \xi, \xi' + A\xi^{\perp}\rangle)\,. \end{aligned}$$

Finally, since the projections

$$\mathcal{N}_G \times_{i(G^*)} \mathcal{N}_{\mathbb{F}}^{\perp} \to \mathcal{N}_G / i(G^*) \quad \text{and}$$

$$F^{\perp} = \{(\xi, v) \in \mathbb{F}^{n-1} \times \mathrm{Im}\,\mathbb{F}\,:\ \xi \in P(\mathcal{N}_G) \oplus \mathcal{N}_{\mathbb{F}}^{\perp}\}/G^* \to \mathcal{N}_G/G^*$$

are the same, we prove that $F^{\perp}$ is a fiber product, $F^{\perp} = \mathcal{N}_G \times_{i(G^*)} \mathcal{N}_{\mathbb{F}}^{\perp}$.

Now to finish the proof by using Lemma 3, we only need to show that the homomorphism $\rho : i(G^*) \to GL(\mathcal{N}_{\mathbb{F}}^{\perp})$ can be extended to a homomorphism $\overline{\rho}$: $\mathcal{N}_G \to GL(\mathcal{N}_{\mathbb{F}}^{\perp})$. Due to our choice of the finite index subgroup $G^* \subset G$, we have that its rotational part is contained in a toral subgroup T of the compact linear group C of automorphisms of $\mathcal{N} = \mathbb{F}^{n-1} \times \mathrm{Im}\,\mathbb{F}$ (either $U(n-1)$ if $\mathbb{F} = \mathbb{C}$, or $Sp(n-1)$ if $\mathbb{F} = \mathbb{H}$, or $Spin(7)$ if $\mathbb{F} = \mathbb{O}$) which pointwise fixes $\mathcal{N}_{\mathbb{F}}$. It shows that $T \subset C_{\mathcal{N}_{\mathbb{F}}^{\perp}} \subset GL(\mathcal{N}_{\mathbb{F}}^{\perp})$. Let $\mathfrak{c} = \mathfrak{c}_{\mathcal{N}_{\mathbb{F}}^{\perp}}$ and $\mathfrak{t}$ be the Lie algebras of the groups $C_{\mathcal{N}_{\mathbb{F}}^{\perp}}$ and T respectively, and $\exp : \mathfrak{c}_{\mathcal{N}_{\mathbb{F}}^{\perp}} \to C_{\mathcal{N}_{\mathbb{F}}^{\perp}}$ the exponential map. Since T is a torus, $\mathfrak{t}$ is an Abelian Lie algebra and $\exp|_{\mathfrak{t}} : \mathfrak{t} \to T$ is a homomorphism.

Let us first assume that $\mathcal{N}_G$ is Abelian. Then $i(G^*)$ is free Abelian as a lattice in $\mathcal{N}_G$. Suppose $\{(\xi_1, v_1), \dots, (\xi_k, v_k)\}$ is a basis for this free Abelian group. Then $\{(\xi_1, v_1), \dots, (\xi_k, v_k)\}$ is also a basis for the vector space $\mathcal{N}_G$. Let $A_1, \dots, A_k \in \mathfrak{t}$ be such that $\exp(A_i) = \rho(\xi_i, v_i)$ for $i = 1, \dots, k$.

Now we can define $\overline{\rho} :\ \mathcal{N}_G \longrightarrow C_{\mathcal{N}_{\mathbb{F}}^{\perp}}$ as the following:

$$\bar{\rho}(t_1\xi_1 + \cdots + t_k\xi_k,\ t_1 v_1 + \cdots + t_k v_k) \;=\; \exp(t_1 A_1 + \cdots + t_k A_k)\,, \tag{3.10}$$

where $t_1, \dots, t_k \in \mathbb{R}$.

Since $\{(\xi_1, v_1), \dots, (\xi_k, v_k)\}$ is a basis for $\mathcal{N}_G$, this map $\bar{\rho}$ is well defined. Moreover, $\overline{\rho}$ is a homomorphism because $\exp|_{\mathfrak{t}} : \mathfrak{t} \to T$ is a homomorphism. So, in the Abelian case, the homomorphism $\bar{\rho}$ clearly extends ρ.

In the second case of non-Abelian $\mathcal{N}_G$, the group $i(G^*)$ is non-Abelian as a lattice in $\mathcal{N}_G$. Thus $i(G^*)$ contains vertical translations, that is, elements of the form $(0, v)$ with $v \neq 0$. Let K be the subgroup of $i(G^*)$ consisting of vertical translations. Then we have the following exact sequence:

$$1 \to K \to i(G^*) \to i(G^*)/K \to 1\,. \tag{3.11}$$

In this sequence, $i(G^*)/K$ is a finitely generated Abelian group because the group $i(G^*)$ is finitely generated and its commutator subgroup consists of vertical translations. Moreover, $i(G^*)/K$ is also torsion free and hence free Abelian, because a nonzero multiple of a non-vertical translation can not be vertical.

Let us consider elements $(\xi_1, v_1), \dots, (\xi_{k-1}, v_{k-1}) \in i(G^*)$ representing those classes in $i(G^*)/K$ that provide a basis for the free Abelian group $i(G^*)/K$, and let $G_0 \subset G^*$ be the subgroup whose image, $i(G_0)$, is generated by these elements:

$i(G_0) = \langle(\xi_1, v_1), \ldots, (\xi_{k-1}, v_{k-1})\rangle$. Due to the above exact sequence (3.11), the group $i(G_0)$ can not be Abelian because $i(G^*)$ is not Abelian. Thus $i(G_0)$ contains vertical translations and $K \cap i(G_0)$ is non-trivial. From this and (3.11), we have the following exact sequence:

$$1 \to K \cap i(G_0) \to i(G_0) \to i(G^*)/K \to 1\,,$$

where K is infinite free Abelian, and $K \cap i(G_0)$ is a finite index subgroup of K. This implies that the group $i(G_0)$ is a finite index subgroup of $i(G^*)$. So now we can take G_0 as the desired finite index subgroup of the group G.

Let $A_1, \ldots, A_{k-1}$ be those elements of the Lie algebra $\mathfrak{t}$ whose images $\exp(A_i)$ equal to the images $\rho(\xi_i, v_i)$ of the above elements $(\xi_i, v_i) \in i(G^*)$ under the homomorphism ρ in (3.7). Then we can define the desired extension $\bar\rho$ of $\rho|_{i(G_0)}$,

$$\bar\rho\,:\; \mathcal{N}_G \longrightarrow U(\mathcal{N}_{\mathbb{F}}^{\perp})\,,$$

in the following way:

$$\bar\rho(t_1\xi_1 + \cdots + t_{k-1}\xi_{k-1},\; v) = \exp(t_1 A_1 + \cdots + t_{k-1}A_{k-1})\,, \tag{3.12}$$

where $t_1, \ldots, t_{k-1} \in \mathbb{R}$.

Indeed, this map $\bar\rho$ is a well defined homomorphism. It follows from two facts: firstly, the elements $\xi_1, \ldots, \xi_{k-1}$ form a basis for $P(\mathcal{N}_G)$, and secondly, the exponential map, $\exp|_{\mathfrak{t}}\,:\,\mathfrak{t} \to T$, is a homomorphism. This homomorphism $\bar\rho$ extends the original homomorphism $\rho|_{i(G_0)}$, $\bar\rho|_{i(G_0)} = \rho|_{i(G_0)}$, because these homomorphisms agree on the generators of $i(G_0)$.

Therefore, in both cases of Abelian and non-Abelian $\mathcal{N}_G$, Lemma 3 and extensions (3.10) and (3.12) of the homomorphism (3.7) complete the proof of triviality of the vector bundle $F^{\perp}$ in (3.6) and hence that of the vector bundle $\mathcal{N}/G_0 \to \mathcal{N}_G/G_0$, provided $\{0\} \times \mathbb{R} \subset \mathcal{N}_G$.

In the remaining case $\{0\} \times \mathbb{R} \not\subseteq \mathcal{N}_G$, our G-invariant Lie subgroup $\mathcal{N}_G$ must be Abelian, and the proof is quite similar to the above. So finally, for a finite index subgroup $G_0 \subset G$, the vector bundle $\mathcal{N}/G$ is finitely covered by the trivial bundle $\mathcal{N}/G_0 \to \mathcal{N}_G/G_0$ which is the product of a compact nilmanifold $\mathcal{N}_G/G_0$ and an Euclidean space. $\square$

Acknowledgements. Supported in part by OU Faculty Senate Award and Centre de Recerca Matematica at Barcelona

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