DIFFERENTIAL GEOMETRY AND ITS APPLICATIONS
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Differential correspondences and characteristics

R. Alonso, S. Jiménez, J.Rodríguez

Abstract. This paper shows how the theory of differential correspondences (also called Lie correspondences) allows giving a unifying approach to the study of characteristics of a system of partial differential equations. We characterize the tangent space to a Lie correspondence on a smooth manifold M at a point in terms of derivations from $C^\infty(M)$ into a suitable Weil algebra and define the characteristic jets which give, as particular cases, the classical Cauchy and Monge characteristics, as well as the characteristic co-vectors.

Introduction

It is well known the important role played by the characteristics of a system of partial differential equations regarding its integration; this is the reason why they have been deeply studied from different points of view since the last years of the 19th century. Several matematicians like Cauchy, Monge, Darboux or Lie developed methods of integration based in the concept of characteristic (see [**6, 14**] and the references there). More recently some other attempts to define the concept of characteristic appeared in the literature, mainly in the framework of Cartan's differential systems: characteristic forms, characteristic cohomology, etc. (see [**4, 5, 9, 11, 19**]).

This paper is a first step in the development of a unifying approach to all the usual definitions of characteristics of a SPDE in the framework of Weil jets. Our work has its origin in a paper published in a 1895 Lie's paper [**14**]; after giving a survey of the most important theories of integration at that time, such as the ones by Darboux or Lévy, he suggests that it is possible to come to a better understanding of what the characteristics are by associating with each SPDE a first order system with only one unknown function. This is done by means of certain canonical correspondences between jet spaces.

For a better understanding of Lie's ideas it is convenient to think of jets of a smooth manifold as ideals of its ring of smooth functions. This point of view, which was introduced in [**15**], is a natural continuation of Weil's theory of near points [**20**] and it allows describing the process of prolongation, the affine structures, the contact system, etc., in terms of the ring of smooth functions of the original manifold, making the fibration unnecessary and simplifying essentially the calculus in local coordinates. Some applications of this theory to different topics can be found in [**2, 16, 17, 18**].

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Following the scheme outlined in [**14**] and within the framework of the above mentioned theory of jets, in [**7, 8**] a new theory of differential correspondences is developed, with applications to systems of partial differential equations, that clarifies and completes some of the partial results announced by Lie. A point of view very close to this one allowed the first author to give the first intrinsic formulation and proof of Drach's theorem in [**2**].

In this paper we will show how the theory of differential correspondences is an adequate framework to give a definition of characteristic which contains the most common approaches to this subject.

In the first section we compile the main topics about jets and partial differential equations in the form they will be used later on; since we will use jets of submanifolds only, we focus our attention in them, though most of the results are generalizable to A-jets, where A is an arbitrary Weil algebra.

In Section 2 we resume the theory of differential correspondences, which are defined by means of the relation of inclusion of ideals, following [**7**]. Ehresmann's point of view is not appropriate for this theory, because our correspondences involve jets of different dimensions; when the manifold is fibred over a fixed base manifold all the jets have the same dimension, and hence the correspondences cannot be established. We give a new description of the equations of the Lie correspondence suitable for a generalization to higher order jets and characterize the tangent space to a differential correspondence at a point in a way close to the characterization of the tangent space to the jet manifolds given in [**15**].

In the third section we define the characteristic jets of a system of partial differential equations. We show, in the examples of the last section, how this definition gives, in particular, the Cauchy and Monge characteristics as well as the characteristic co-vectors, whose relationship with the Cauchy-Kowalevsky theorem was shown in [**17**].

Notations and conventions. In the entirety of this paper M will be an n-dimensional smooth manifold and the word 'submanifold' will mean 'locally closed submanifold.' When X is a locally closed submanifold of M, I_X will be the ideal of $C^\infty(M)$ consisting of the functions vanishing on X. In case of X being only locally closed, we should replace M by the open set U into which X is a closed submanifold. Nevertheless, for simplicity in the exposition, that will be implicitly understood.

1. Jets of submanifolds

Our theory of differential correspondences is based in the presentation of jet spaces given in [**15**], which is a natural continuation of the theory of Weil's near points [**20**]. A near point of a smooth manifold M is a homomorphism of $\mathbb{R}$-algebras from $C^\infty(M)$ into a Weil algebra; according our definition, a jet of M is the kernel of a near point of M in the sense of Weil, and consequently it is an ideal of $C^\infty(M)$. The manifolds of jets defined in this way include Grassmann manifolds or jets of submanifolds as particular cases. A detailed exposition of this theory can be found in [**1, 15**]. Here we will use jets of submanifolds only; next we will give the basic definitions and summarize, without proofs, some results which will be used later on.

Let M be an n-dimensional smooth manifold; if X is an m-dimensional submanifold of M, for each point $p \in X$ we define the (m, ℓ)-jet of X at p as the class of all m-dimensional submanifolds of M which have a contact of order ℓ with X at p. We can associate with its (m,ℓ)-jet at p the ideal $\mathfrak{p}_m^\ell = I_X + \mathfrak{m}_p^{\ell+1}$. This gives a bijection between the set of (m,ℓ)-jets of M and the set of ideals $\mathfrak{p}_m^\ell$ of $C^\infty(M)$ such that the factor ring $C^\infty(M)\big/\mathfrak{p}_m^\ell$ is isomorphic to the Weil algebra $\mathbb{R}_m^\ell$ of polynomials of degree $\leqslant \ell$ in m variables (in fact, they are the kernels of the regular (m,ℓ)-velocities of M). When $m = \dim M$, each (m,ℓ)-jet of M has the form $\mathfrak{m}_p^{\ell+1}$, where $p \in M$.

We will denote by $J_m^\ell M$ the set of all (m,ℓ)-jets of M. There is a canonical projection $\pi^\ell \colon J_m^\ell M \longrightarrow M$ which assigns to each jet $\mathfrak{p}_m^\ell$ the unique maximal ideal $\mathfrak{p}_m^0 = \mathfrak{m}_p$ of $C^\infty(M)$ containing $\mathfrak{p}_m^\ell$; the point $p \in M$ corresponding to this ideal is called the source of $\mathfrak{p}_m^\ell$.

Remark 1. *When $\ell = 1$ we can give a geometric description of jets: there is a bijection between the $(m,1)$-jets of M with source p and the m-dimensional subspaces of T_pM. In fact, we can associate with each first order jet $\mathfrak{p}_m^1 \in J_m^1 M$ the vector space*

$$L_{\mathfrak{p}_m^1} = \{D_p \in T_pM : D_p(f) = 0 \ \forall f \in \mathfrak{p}_m^1\}$$

and conversely, the subspace $L \subseteq T_pM$ defines the jet

$$\mathfrak{p}_m^1 = \{f \in C^\infty(M) : D_p(f) = 0 \ \forall D_p \in L\}.$$

Therefore $J_m^1 M$ is the Grassmann manifold of m-planes tangent to M.

In **[15]** $J_m^\ell M$ is endowed with a smooth structure as a quotient of the space of regular (m,ℓ)-velocities of M by the action of the differential group $\mathrm{Aut}(\mathbb{R}_m^\ell)$. A system of local coordinates may be described as follows:

Let $\mathfrak{p}_m^\ell \in J_m^\ell M$; if X is an m-dimensional submanifold of M whose ℓ-jet is $\mathfrak{p}_m^\ell$, we can find a neighbourhood of p coordinated by functions $x_1, \dots, x_m, y_1, \dots, y_{n-m}$ such that the local equations of X can be written in the form

$$y_j = f_j(x_1, \dots, x_m); \qquad (1 \leqslant j \leqslant n-m)$$

then $\mathfrak{p}_m^\ell$ is the sum of $\mathfrak{m}_p^{\ell+1}$ and the ideal spanned by the $n-m$ functions

$$y_j - \sum_{|\alpha| \leqslant \ell} \frac{1}{\alpha!} \frac{\partial^{|\alpha|} f_j}{\partial x^\alpha}(p) \ (x - x(p))^\alpha,$$

where $\alpha = (\alpha_1, \dots, \alpha_m)$ is a multi-index and

$$(x - x(p))^\alpha = (x_1 - x_1(p))^{\alpha_1} \cdot \ldots \cdot (x_m - x_m(p))^{\alpha_m}.$$

The functions $x_i, y_{j,\alpha}$ $(1 \leqslant i \leqslant m,\ 1 \leqslant j \leqslant n-m,\ |\alpha| \leqslant \ell)$ defined by

$$x_i(\mathfrak{p}_m^\ell) = x_i(p), \qquad y_{j,\alpha}(\mathfrak{p}_m^\ell) = \frac{\partial^{|\alpha|} f_j}{\partial x^\alpha}(p)$$

are local coordinates in an open subset of $J_m^\ell M$ (note that $y_{j,0} = y_j$).

The above coordinates can be obtained, when jets are thought of as ideals, without any reference to the submanifold X, in the following way: when $\mathfrak{p}_m^\ell$ is the jet at p of an m-dimensional submanifold of M parametrized locally by $x_1, \dots, x_m$, it is said to be proper with respect to the ring $\mathbb{R}[x_1, \dots, x_m]$. This condition means that the ring $C^\infty(M)\big/\mathfrak{p}_m^\ell$ is isomorphic to the algebra of polynomials

$$\mathbb{R}_m^\ell = \mathbb{R}[x_1, \dots, x_m] \Big/ (x_1 - x_1(p), \dots, x_m - x_m(p))^{\ell+1},$$

hence for each function $f \in C^\infty(M)$ there is a unique polynomial $P_f \in \mathbb{R}_m^\ell$ such that $f - P_f \in \mathfrak{p}_m^\ell$; the coefficient of $(x - x(p))^\alpha$ in the polynomial associated with y_j is

$y_{j,\alpha}(\mathfrak{p}^\ell_m)$. The set of the jets proper with respect to $\mathbb{R}[x_1,\dots,x_m]$ is an open subset coordinated by the functions

$$x_i, y_{j,\alpha}, \qquad (1 \leqslant i \leqslant m, 1 \leqslant j \leqslant n-m, |\alpha| \leqslant \ell).$$

This open subset is isomorphic to the space of ℓ-jets of sections of the projection $(x_i, y_j) \longmapsto x_i$. This is the reason why we use the notations x_i, y_j, thus establishing a distinction between the "base coordinates" and the "fibre coordinates". Nevertheless, such a distinction is only formal, because m arbitrary coordinates can be chosen as independent variables and the remainder as functions of them.

Remark 2. *This way we see that our theory recovers the original point of view of Lie about jets (see* **[13]***, for instance); he used to consider a system of local coordinates $x_1,\dots,x_n$ in an open subset of M, and to think of m of them as independent and the remainder as dependent variables; but in a dynamical way, without fixing them.*

A fundamental advantage of our theory of jets is the fact that the prolongations of submanifolds or ideals, the tangent structures and other constructions related to the jet spaces of a manifold M can be described in terms of the ring $C^\infty(M)$; we do not need to change the ring of functions, but only the algebra where our mappings or derivations are valued. An important example of this approach is the following characterization of the tangent space $T_{\mathfrak{p}^\ell_m} J^\ell_m M$ given in **[15]** (see **[1]** for the generalization to A-jets).

Theorem 1. *Let M be a smooth manifold; for each jet $\mathfrak{p}^\ell_m \in J^\ell_m M$ the tangent space $T_{\mathfrak{p}^\ell_m} J^\ell_m M$ is canonically isomorphic to the vector space $\mathcal{D}_{\mathfrak{p}^\ell_m}/\mathcal{D}'_{\mathfrak{p}^\ell_m}$, where*

$$\mathcal{D}_{\mathfrak{p}^\ell_m} = \mathrm{Der}_{\mathbb{R}}\left(C^\infty(M), C^\infty(M)\Big/\mathfrak{p}^\ell_m\right)$$

and $\mathcal{D}'_{\mathfrak{p}^\ell_m}$ is the subspace of $\mathcal{D}_{\mathfrak{p}^\ell_m}$ containing the derivations which annihilate $\mathfrak{p}^\ell_m$.

In the local coordinates described above, this identification gives:

$$\begin{aligned} T_{\mathfrak{p}^\ell_m} J^\ell_m M &\longrightarrow \mathcal{D}_{\mathfrak{p}^\ell_m}/\mathcal{D}'_{\mathfrak{p}^\ell_m} \\ \left(\frac{\partial}{\partial x_i}\right)_{\mathfrak{p}^\ell_m} &\longmapsto \left[\frac{\partial}{\partial x_i}\right] \\ \left(\frac{\partial}{\partial y_{j,\alpha}}\right)_{\mathfrak{p}^\ell_m} &\longmapsto \left[\frac{(x-x(p))^\alpha}{\alpha!}\frac{\partial}{\partial y_j}\right] \end{aligned}$$

where we write $[D]$ for the class of the derivation D.

Let X be an m-dimensional submanifold of M; then $J^\ell_m X$ is diffeomorphic to a submanifold of $J^\ell_m M$, called the prolongation of X to $J^\ell_m M$. Its points are the jets of the form $I_X + \mathfrak{m}_p^{\ell+1}$, where $p \in X$. It is easy to see that

$$J^\ell_m X = \{\mathfrak{p}^\ell_m \in J^\ell_m M : \mathfrak{p}^\ell_m \supseteq I_X\}.$$

In the above local coordinates, if the equations of X are

$$y_j - f_j(x_1,\dots,x_m) = 0 \qquad (1 \leqslant j \leqslant n-m)$$

then the local equations of its prolongation are

$$\begin{aligned} y_j &= f_j(x_1,\dots,x_m), && (1 \leqslant j \leqslant n-m) \\ y_{j,\alpha} &= \frac{\partial^{|\alpha|} f_j}{\partial x^\alpha} && (1 \leqslant j \leqslant n-m; 1 \leqslant |\alpha| \leqslant \ell) \end{aligned}$$

An immediate consequence of the definitions is the following result:

PROPOSITION 1. *If X is an m-dimensional submanifold of M, for each $\mathfrak{p}_m^\ell \in J_m^\ell X$ the tangent space $T_{\mathfrak{p}_m^\ell} J_m^\ell X$ is isomorphic to the set of classes of derivations in $\mathcal{D}_{\mathfrak{p}_m^\ell}/\mathcal{D}'_{\mathfrak{p}_m^\ell}$ killing I_X.*

There is a Pfaff system on $J_m^\ell M$, called the contact or Cartan system, which vanish on the prolongations to $J_m^\ell M$ of the m-dimensional submanifolds of M. Theorem 1 allows giving an intrinsic definition of the contact system, whose value at each jet $\mathfrak{p}_m^\ell$ consists of certain differentials of the functions belonging to $\mathfrak{p}_m^\ell$ (see [**3**], where the contact system is defined for general A-jets). In the above coordinates, the contact system is generated by the 1-forms

$$\omega_{j,\alpha} = dy_{j,\alpha} - \sum_{i=1}^{m} dy_{j,\alpha+1_i}\, dx_i, \qquad (1 \leqslant i \leqslant m, |\alpha| \leqslant \ell - 1),$$

where 1_i is the m–index with 1 in the ith component and 0 in the remaining ones. Its associated distribution of vector fields is spanned by:

$$\partial_i^{(\ell)} = \frac{\partial}{\partial x_i} + \sum_{\substack{|\alpha|=\ell-1\\ 1\leqslant j\leqslant n-m}} y_{j,\alpha+1_i} \frac{\partial}{\partial y_{j,\alpha}} \qquad (1 \leqslant i \leqslant m)$$

$$\frac{\partial}{\partial y_{j,\beta}} \qquad (1 \leqslant j \leqslant n - m,\ |\beta| = \ell).$$

DEFINITION 1. *A system of partial differential equations of order ℓ, in m independent variables, over the manifold M, is a locally closed submanifold $\mathcal{R}_m^\ell$ of $J_m^\ell M$. A classical solution of $\mathcal{R}_m^\ell$ is a m-dimensional submanifold $X \subseteq M$ such that $J_m^\ell X \subseteq \mathcal{R}_m^\ell$. A generalized solution is an m–dimensional submanifold $\overline{X} \subseteq J_m^\ell M$ solution of the contact system such that $\overline{X} \subseteq \mathcal{R}_m^\ell$.*

The prolongation $\mathcal{R}_m^{\ell+1}$ of a ℓ–th order PDE system $\mathcal{R}_m^\ell$ to order $\ell + 1$ is made via the canonical immersion $J_m^{\ell+1}M \hookrightarrow J_m^1 J_m^\ell M$. More precisely,

$$\mathcal{R}_m^{\ell+1} = \left\{\mathfrak{p}_m^{\ell+1} \ :\ \mathfrak{p}_m^{\ell+1} \supseteq I(\mathcal{R}_m^\ell)\right\},$$

where $I(\mathcal{R}_m^\ell)$ is the ideal of $\mathcal{R}_m^\ell$ in $C^\infty(J_m^\ell M)$ and the inclusion is as ideals of $C^\infty(J_m^\ell M)$.

If $\mathcal{R}_m^\ell$ is defined locally by the equations

$$\varphi_k(x_i, y_j, y_{j,\alpha}) = 0 \qquad (1 \leqslant k \leqslant s)$$

the local equations of $\mathcal{R}_m^{\ell+1}$ are

$$\begin{aligned} \varphi_k(x_i, y_j, y_{j,\alpha}) &= 0, \\ \partial_i^{(\ell+1)}\ \varphi_k &= 0, \end{aligned} \qquad (1 \leqslant k \leqslant s, 1 \leqslant i \leqslant m)$$

2. Differential correspondences

Since each jet in M is an ideal of $C^\infty(M)$, the relation of inclusion between ideals gives canonical correspondences between different jet spaces. Next we define the correspondences for first order jets following [**7, 8**], though most of the constructions and results can be generalized for higher orders.

DEFINITION 2. *Given two integers r, m such that $0 \leqslant r \leqslant m \leqslant n$, the Lie correspondence $\bigwedge_{m,r} = \bigwedge_{m,r}(M)$ is the subset of the fibred product $J_m^1 M \times_M J_r^1 M$*

consisting of the pairs of jets $(\mathfrak{p}^1_m, \mathfrak{q}^1_r)$ *such that* $\mathfrak{p}^1_m \subseteq \mathfrak{q}^1_r$ *(inclusion as ideals of* $C^\infty(M)$*).*

A geometric interpretation of these correspondences results from thinking of each first order jet over $p \in M$ as a linear subspace of T_pM: in the notations of Remark 1, the inclusion between the ideals $\mathfrak{p}^1_m \subseteq \mathfrak{q}^1_r$ becomes an inclusion $L_{\mathfrak{p}^1_m} \supseteq L_{\mathfrak{q}^1_r}$ between linear subspaces of T_pM.

The Lie correspondence $\bigwedge_{m,r}$ is endowed with the two natural projections, π_1 and π_2, such that the diagram

$$\begin{tikzcd} & \bigwedge_{m,r} \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ J^1_m M \arrow[dr] & & J^1_r M \arrow[dl] \\ & M & \end{tikzcd}$$

is commutative.

Remark 3. *At this point it is essential to stop thinking of jets as 'jets of cross-sections of a fibred manifold', because when M is fibred over a manifold X, all the jets have the same dimension (*$= \dim X$*) and the above correspondences cannot be established.*

In **[3]** it is pointed out that the value of the contact system at a jet is essentially the jet itself; this idea was used in **[7, 8]** to characterize the Lie correspondence by means of inclusions between contact systems and to obtain its local equations. Here we give another description of the local equations of the Lie correspondence in a straightforward way, suitable for a generalization to higher orders.

Each jet $\mathfrak{p}^1_m$ gives rise to a canonical map $\varphi_{\mathfrak{p}^1_m} : C^\infty(M) \longrightarrow C^\infty(M) \big/ \mathfrak{p}^1_m$ If $\mathfrak{p}^1_m \subseteq \mathfrak{q}^1_r$, then we have a commutative diagram

$$\begin{tikzcd} C^\infty(M) \arrow[r, "\varphi_{\mathfrak{p}^1_m}"] \arrow[dr, "\varphi_{\mathfrak{q}^1_r}"'] & C^\infty(M) \big/ \mathfrak{p}^1_m \arrow[d, "\pi"] \\ & C^\infty(M) \big/ \mathfrak{q}^1_r \end{tikzcd} \tag{1}$$

where π is the projection induced by $\varphi_{\mathfrak{q}^1_r}$. Then the equations of the correspondence are

$$\varphi_{\mathfrak{q}^1_r}(f) = (\pi \circ \varphi_{\mathfrak{p}^1_m})(f), \qquad \forall f \in C^\infty(M). \tag{2}$$

These equations must be understood as equalities between polynomials, after the quotient rings $C^\infty(M) \big/ \mathfrak{p}^1_m$ and $C^\infty(M) \big/ \mathfrak{q}^1_r$ are identified with the rings of polynomials $\mathbb{R}^1_m$ and $\mathbb{R}^1_r$, respectively, as explained in Section 1.

Let $x_1, \dots, x_m, y_1, \dots, y_{n-m}$ be a system of local coordinates on an open subset U of M. In the set of $(m,1)$-jets whose projection lies in U and that are proper with respect to the ring $\mathbb{R}[x_1, \dots, x_m]$ we have the usual coordinates

$$x_h, y_j, y_{j,h}, \qquad (1 \leqslant h \leqslant m, 1 \leqslant j \leqslant n-m).$$

In a similar way, the set of the jets proper with respect to $\mathbb{R}[x_1, \dots, x_r]$ is coordinated by the functions

$$x_i, x_k, y_j, x_{k,i}, \overline{y}_{j,i}, \qquad (1 \leqslant i \leqslant r, r+1 \leqslant k \leqslant m, 1 \leqslant j \leqslant n-m),$$

where the $\overline{y}_{j,i}$ have the same meaning than the $y_{j,h}$ above.

If in Diagram (1) we identify $C^\infty(M)\big/\mathfrak{p}^1_m$ with the ring of polynomials $\mathbb{R}^1_m$ through the coordinates $x_1,\ldots,x_m$, $C^\infty(M)\big/\mathfrak{q}^1_r$ with $\mathbb{R}^1_r$ through the coordinates $x_1,\ldots,x_r$ (see Section 1) and compute the images of the coordinates y_j, Equations (2) give

$$\overline{y}_{j,i}-y_{j,i}-\sum_{k=r+1}^{m} y_{j,k}x_{k,i}=0, \qquad (1\leqslant j\leqslant n-m, 1\leqslant i\leqslant r) \tag{3}$$

which are the local equations of $\bigwedge_{m,r}$.

For each PDE system $\mathcal{R}^1_m\subseteq J^1_mM$, $\mathcal{R}^1_{m,r}$ will denote the intersection of $\bigwedge_{m,r}$ with $\mathcal{R}^1_m\times_M J^1_r(M)$. The local equations of the correspondence $\mathcal{R}^1_{m,r}$ as a submanifold of $J^\ell_mM\times_M J^1_r(M)$ are obtained by adding to Equations (3) the local equations of $\mathcal{R}^1_m$.

We have the commutative diagram

$$\begin{tikzcd} & \mathcal{R}^1_{m,r} \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ \mathcal{R}^1_m \arrow[dr] & & \overline{\mathcal{R}}^1_r\subseteq J^1_rM \arrow[dl] \\ & M & \end{tikzcd}$$

where $\overline{\mathcal{R}}^1_r=\pi_2(\mathcal{R}^1_{m,r})$; $\overline{\mathcal{R}}^1_r$ is a SPDE with r independent variables and $n-r$ unknown functions.

If $\mathcal{R}^\ell_m$ is a SPDE of order ℓ, it can be considered as a SPDE of first order via the natural immersion

$$J^\ell_mM\hookrightarrow J^1_m(J^{\ell-1}_mM)$$

Hence the theory of correspondences can be applied to those systems (here the base-manifold is $J^{\ell-1}_mM$ instead of M). If we write

$$\mathcal{R}^\ell_{m,r}=\bigwedge\nolimits_{m,r}(J^{\ell-1}_mM)\bigcap\left[\mathcal{R}^\ell_m\times_{J^{\ell-1}_mM}J^1_r(J^{\ell-1}_mM)\right],$$

we have the following commutative diagram:

$$\begin{tikzcd} & \mathcal{R}^\ell_{m,r} \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ \mathcal{R}^\ell_m \arrow[dr] & & \overline{\mathcal{R}}^1_r\subseteq J^1_r(J^{\ell-1}_mM) \arrow[dl] \\ & J^{\ell-1}_mM & \end{tikzcd}$$

Remark 4. *In* [**8**] *this kind of correspondences have been applied to show how the integration of a class of involutive PDE systems can be reduced to the integration of first order systems with a single unknown function.*

Our next task is to compute the tangent space to $\bigwedge_{m,r}$ at a couple $(\mathfrak{p}^1_m,\mathfrak{q}^1_r)$; to this end we use the isomorphism between the tangent space $T_{\mathfrak{p}^1_m}J^1_mM$ and the vector space $\mathcal{D}_{\mathfrak{p}^\ell_m}/\mathcal{D}'_{\mathfrak{p}^\ell_m}$ given by Theorem 1.

Let $D \in \mathrm{Der}_{\mathbb{R}}\left(C^\infty(M), C^\infty(M)\big/\mathfrak{p}^1_m\right)$ and let us denote by $\widetilde{D}$ the derivation from $C^\infty(M)$ into $C^\infty(M)\big/\mathfrak{q}^1_r$ obtained as the composition of D with the natural projection $\pi\colon C^\infty(M)\big/\mathfrak{p}^1_m \longrightarrow C^\infty(M)\big/\mathfrak{q}^1_r$; we have the commutative diagram

$$\begin{tikzcd} C^\infty(M) \arrow[r, "D"] \arrow[dr, "\widetilde{D}"'] & C^\infty(M)\big/\mathfrak{p}^1_m \arrow[d, "\pi"] \\ & C^\infty(M)\big/\mathfrak{q}^1_r \end{tikzcd}$$

(Note that the class of $\widetilde{D}$ in $T_{\mathfrak{q}^1_r} J^1_r M$ depends on the derivation D). We have:

Theorem 2. *The tangent space $T_{(\mathfrak{p}^1_m,\mathfrak{q}^1_r)}\bigwedge_{m,r}$ is canonically isomorphic of the vector space of all (fibred) sums $[D]+[\widetilde{D}]$, where $[D]$ is the class of a derivation D from $C^\infty(M)$ into $C^\infty(M)\big/\mathfrak{p}^1_m$ and $\widetilde{D}=\pi\circ D$.*

Proof. If $(\mathfrak{p}^1_m,\mathfrak{q}^1_r)\in\bigwedge_{m,r}$, it is possible to choose a system of local coordinates

$$x_1,\dots,x_r,x_{r+1},\dots,x_m,y_1,\dots,y_{n-m}$$

on a neighbourhood of $p=\mathfrak{p}^0_m$ in M which vanish at p such that

$$\mathfrak{p}^1_m=(y_1,\dots,y_{n-m})+\mathfrak{m}^2_p,\quad \mathfrak{q}^1_r=(x_{r+1},\dots,x_m,y_1,\dots,y_{n-m})+\mathfrak{m}^2_p;$$

we endow a neighbourhood of $(\mathfrak{p}^1_m,\mathfrak{q}^1_r)$ in $J^1_mM\times_M J^1_rM$ with the system of local coordinates

$$x_i,x_k,y_j,y_{j,i},y_{j,k},x_{k,i},\overline{y}_{j,i},$$

where $1\leqslant i\leqslant r, r+1\leqslant k\leqslant m, 1\leqslant j\leqslant n-m$. If we identify $C^\infty(M)\big/\mathfrak{p}^1_m$ and $C^\infty(M)\big/\mathfrak{q}^1_r$ with the rings of polynomials $\mathbb{R}[x_1,\dots,x_m]\big/(x_1,\dots,x_m)^2$ and $\mathbb{R}[x_1,\dots,x_r]\big/(x_1,\dots,x_r)^2$, respectively, then the derivations

$$\frac{\partial}{\partial x_i},\frac{\partial}{\partial x_k},\frac{\partial}{\partial y_j},x_i\frac{\partial}{\partial y_j},x_k\frac{\partial}{\partial y_j},x_i\frac{\partial}{\partial x_k}$$

give rise to the values at $(\mathfrak{p}^1_m,\mathfrak{q}^1_r)$ of the tangent vector fields

$$\frac{\partial}{\partial x_i},\frac{\partial}{\partial x_k},\frac{\partial}{\partial y_j},\frac{\partial}{\partial y_{j,i}}+\frac{\partial}{\partial \overline{y}_{j,i}},\frac{\partial}{\partial y_{j,k}},\frac{\partial}{\partial x_{k,i}}.$$

Those vectors span the tangent space to $\bigwedge_{m,r}$ at $(\mathfrak{p}^1_m,\mathfrak{q}^1_r)$. □

Corollary 1. *The subspace of $T_{(\mathfrak{p}^1_m,\mathfrak{q}^1_r)}\bigwedge_{m,r}$ vertical for the projection*

$$\pi_2\colon\bigwedge\nolimits_{m,r}\longrightarrow J^1_mM$$

is the set of classes of derivations from $C^\infty(M)$ into $C^\infty(M)\big/\mathfrak{p}^1_m$ which value in $\mathfrak{q}^1_r\big/\mathfrak{p}^1_m$.

3. Differential correspondences and characteristics

In this section we describe the characteristics of a system of PDE using the framework of differential correspondences.

Let us denote by

$$\bigwedge\nolimits^{\ell}_{m,r}=\bigwedge\nolimits_{m,r}(J^{\ell-1}_mM)\bigcap\left[J^\ell_mM\times_{J^{\ell-1}_mM}J^1_r(J^{\ell-1}_mM)\right],$$

where $J^\ell_m M$ is considered as a submanifold of $J^1_m(J^{\ell-1}_m M)$ via the natural immersion. We have a commutative diagram

$$\begin{tikzcd} & \bigwedge^\ell_{m,r} \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ J^\ell_m M \arrow[dr] & & J^1_r(J^{\ell-1}_m M) \arrow[dl] \\ & J^{\ell-1}_m M & \end{tikzcd}$$

Let $Q_{\mathfrak{p}^\ell_m} J^\ell_m M$ be the tangent space to the fibre of the projection $J^\ell_m M \longrightarrow J^{\ell-1}_m M$ which contains $\mathfrak{p}^\ell_m$. If $(\mathfrak{p}^\ell_m, \mathfrak{q}^1_r) \in \bigwedge^\ell_{m,r}$, the subspace of $T_{(\mathfrak{p}^\ell_m,\mathfrak{q}^1_r)} \bigwedge^\ell_{m,r}$ vertical for the projection π_2 is canonically isomorphic to a subspace $\widetilde{Q}_{(\mathfrak{p}^\ell_m,\mathfrak{q}^1_r)}$ of $Q_{\mathfrak{p}^\ell_m} J^\ell_m M$. Accordingly, if $\mathcal{R}^\ell_m$ is a SPDE, the subspace of $T_{(\mathfrak{p}^\ell_m,\mathfrak{q}^1_r)} \mathcal{R}^\ell_{m,r}$ vertical for the projection π_2 is canonically isomorphic to a subspace $\widetilde{G}_{(\mathfrak{p}^\ell_m,\mathfrak{q}^1_r)}$ of the symbol $G_{\mathfrak{p}^\ell_m}$ of $\mathcal{R}^\ell_m$ at $\mathfrak{p}^\ell_m$; namely,

$$\widetilde{G}_{(\mathfrak{p}^\ell_m,\mathfrak{q}^1_r)} = \widetilde{Q}_{(\mathfrak{p}^\ell_m,\mathfrak{q}^1_r)} \cap G_{\mathfrak{p}^\ell_m}.$$

DEFINITION 3. *Let $(\mathfrak{p}^\ell_m, \mathfrak{q}^1_r) \in \mathcal{R}^\ell_{m,r}$; the jet $\mathfrak{q}^1_r \in J^1_r(J^{\ell-1}_m M)$ is* ***non-characteristic for $\mathcal{R}^\ell_m$ at $\mathfrak{p}^\ell_m$*** *if the subspaces $\widetilde{Q}_{(\mathfrak{p}^\ell_m,\mathfrak{q}^1_r)}$ and $G_{\mathfrak{p}^\ell_m}$ intersect transversally in $Q_{\mathfrak{p}^\ell_m} J^\ell_m M$, and* ***characteristic*** *otherwise.*

REMARK 5. *The characteristic r-dimensional jets represent the infinitesimal r-dimensional submanifolds of M contained in "too much" m-dimensional solutions of $\mathcal{R}^\ell_m$. In particular, the characteristic jets of dimension $m-1$ represent the $(m-1)$-dimensional "infinitesimal submanifolds" up to order $\ell-1$ which are characteristic data for a Cauchy problem.*

We finish this section by computing in local coordinates the condition for a jet to be characteristic for a SPDE $\mathcal{R}^\ell_m$, when $r = m-1$; we assume that the projection $\mathcal{R}^\ell_m \longrightarrow J^{\ell-1}_m M$ is a submersion. The following calculations will be interpreted in the next section in terms of characteristic 1-forms.

Let us take local coordinates

$$x_i, x_m, y_{j,\alpha}, x_{m,i}, \overline{y}_{j,\beta,i},$$

where $(1 \leqslant i \leqslant m-1, 1 \leqslant j \leqslant n-m, 0 \leqslant |\alpha| \leqslant \ell, 0 \leqslant |\beta| \leqslant \ell-1)$, in a neighbourhood of $(\mathfrak{p}^\ell_m, \mathfrak{q}^1_{m-1})$ in $J^\ell_m M \times_{J^{\ell-1}_m M} J^1_{m-1}(J^{\ell-1}_m M)$ such that $\mathfrak{p}^\ell_m = (y_1, \dots, y_{n-m}) + \mathfrak{m}^{\ell+1}_p$ and $\mathfrak{q}^1_{m-1} = \mathfrak{p}^\ell_m + (x_m)$. As an ideal of $C^\infty(J^{\ell-1}_m M)$, $\mathfrak{p}^\ell_m = (y_{j,\alpha}) + \mathfrak{m}^2_{\mathfrak{p}^{\ell-1}_m}$; then the vectors in $\widetilde{Q}_{(\mathfrak{p}^\ell_m,\mathfrak{q}^1_{m-1})}$ are the classes of derivations from $C^\infty(J^{\ell-1}_m M)$ into $C^\infty(J^{\ell-1}_m M) \,/\, \mathfrak{p}^\ell_m$ which are multiple of x_m, that is to say, the vectors of the form

$$[D] = \sum_{\substack{|\alpha|=\ell-1 \\ 1 \leqslant j \leqslant n-m}} \lambda_{j,\alpha} \left(\frac{\partial}{\partial y_{j,\alpha+1_m}} \right)_{\mathfrak{p}^\ell_m}$$

Since those vectors must be tangent to $J^\ell_m M$, the symmetry of the indices implies that

$$[D] = \sum_{1\leqslant j\leqslant n-m} \mu_j \left(\frac{\partial}{\partial y_{j,\ell 1_m}}\right)_{\mathfrak{p}^\ell_m}$$

If $\mathcal{R}^\ell_m$ is defined in a neighbourhood of $\mathfrak{p}^\ell_m$ by equations

$$F_1 = 0, \dots, F_s = 0, \qquad (s \leqslant n-m)$$

the condition for a $\mathfrak{q}^1_r$ to be non-characteristic at $\mathfrak{p}^\ell_m$ is that the rank of the matrix

$$\left(\frac{\partial(F_1,\dots,F_s)}{\partial(y_{1,\ell 1_m},\dots,y_{n-m,\ell 1_m})}\right)_{\mathfrak{p}^\ell_m}$$

equals s; by applying the implicit function theorem it turns out that the system $\mathcal{R}^\ell_m$ is defined, in a neighbourhood of $\mathfrak{p}^\ell_m$, by equations

$$y_{j_k,\ell 1_m} = \varphi_{j_k}(y_j, y_{j,\alpha}) \qquad (1 \leqslant k \leqslant s) \tag{4}$$

If $s = n-m$, then the functions φ_{j_k} do not depend on any variable $y_{j,\ell 1_m}$, $(1 \leqslant j \leqslant n-m)$, hence (4) is the Cauchy-Kowalevski normal form for $\mathcal{R}^\ell_m$ at $\mathfrak{p}^\ell_m$. If the system is underdetermined $(s \leqslant n-m)$, then Equations (4) can be written in the form

$$y_{k,\ell 1_m} = \varphi_k(y_i, y_{j,\alpha}) \qquad (1 \leqslant k \leqslant s) \tag{5}$$

and the functions φ_k may depend on the variables $y_{j,\ell 1_m}$ $(s+1 \leqslant j \leqslant n-m)$, so that Equations (5) are the Cauchy-Kowalevski normal form for $\mathcal{R}^\ell_m$, when it is considered as a system of partial differential equations with $y_1,\dots,y_s$ as unknown functions and $y_{s+1},\dots,y_{n-m}$ as parameters. Its general solution depends on arbitrary functions.

4. Examples

4.1. Characteristic co-vectors. Our first example is the relationship between the characteristic jets and the characteristic co-vectors, defined in [**11, 5, 17**]. The theory of differential correspondences shows how the characteristic 1-forms are a quite restrictive approach, because their definition supposes a fibration, while the characteristic jets can be defined without any fibration on the manifold M.

Let us recall some definitions and results from [**17**] related to this subject.

Let $\mathcal{R}^\ell$ be an ℓ-th order system of partial differential equations on the fibre bundle $\pi\colon M \longrightarrow X$, $(\dim X = m)$; let $\mathfrak{p}^\ell \in \mathcal{R}^\ell$ and suppose that $\mathcal{R}^\ell$ is defined, in a neighbourhood of $\mathfrak{p}^\ell$, by the equations

$$F_1 = 0, \dots, F_s = 0,$$

where $F_1,\dots,F_s \in C^\infty(J^\ell M)$ are functionally independent around $\mathfrak{p}^\ell$.

Let $x \in X$ the source of $\mathfrak{p}^\ell$, $p = \mathfrak{p}^0$, $\omega \in T^*_x X$ and ω^ℓ its ℓ-th symmetric power. Then $\omega^\ell \mathcal{T}_p M$ is a vector subspace of $S^\ell T^*_x X \otimes \mathcal{T}_p M$ which can be thought, via the fundamental identification [**12, 17**], with a subspace of $Q_{\mathfrak{p}^\ell} J^\ell M$, the tangent space to the fibre of the projection $J^\ell M \longrightarrow J^{\ell-1} M$ containing $\mathfrak{p}^\ell$.

Definition 4. *We will say that ω is non-characteristic for $\mathcal{R}^\ell$ at $\mathfrak{p}^\ell$ if the subspaces $\omega^\ell \mathcal{T}_p M$ and $G^\ell_{\mathfrak{p}^\ell}$ intersect transversally in $Q_{\mathfrak{p}^\ell} J^\ell M$, and that it is characteristic in other case.*

The above definition is justified by the following theorem, proved in [**17**]:

THEOREM 3. *If the canonical projection $\mathcal{R}^\ell \longrightarrow J^{\ell-1}M$ is a submersion around $\mathfrak{p}^\ell$, the system $\mathcal{R}^\ell$ can be written in the normal form of Cauchy-Kowalevski in a neighbourhood of $\mathfrak{p}^\ell$ if and only if there is a non-characteristic 1-form for $\mathcal{R}^\ell$ at $\mathfrak{p}^\ell$.*

In the conditions of the computations at the end of the previous section, the open subset of points of $\mathcal{R}^\ell_m$ proper with respect to $\mathbb{R}[x_1,\ldots,x_m]$ is a SPDE over the fibre bundle given by the projection $(x_i,y_j) \longmapsto (x_i)$, and the jet $\mathfrak{q}^1_{m-1}$ is characteristic for $\mathcal{R}^\ell_m$ at $\mathfrak{p}^\ell_m$ if and only if the 1-form $\omega = d_x x_m$ is characteristic at $\mathfrak{p}^\ell_m$.

4.2. Cauchy characteristic field for equation of first order. Let $\dim M = 3$ and consider an open subset of M with coordinates x,y,z. We will use the classical notations, p,q,r,s,t for the derivatives of first and second orders. We will work in an open subset of the jets of $J^1_1(J^1_2 M)$ proper with respect to x; we choose local coordinates $x,y,z,p,q,y_1,z_1,p_1,q_1$ in $J^1_1(J^1_2M)$. For the sake of simplicity, when a tangent vector will appear we will omit the point where the partial derivatives are evaluated.

The Lie correspondence in $J^2_2 M \times_{J^1_2 M} J^1_1(J^1_2M)$ is defined by the equations

$$\begin{aligned} z_1 &= p + q y_1 \\ p_1 &= r + s y_1 \\ q_1 &= s + t y_1 \end{aligned}$$

Let us consider a single PDE

$$\mathcal{R}^1_2 \equiv \{F(x,y,z,p,q) = 0\}.$$

Its first prolongation is the second order system $\mathcal{R}^2_2$ defined by

$$F = 0, \qquad \partial^{(2)}_x F = 0, \qquad \partial^{(2)}_y F = 0,$$

where $\partial^{(2)}_x$ and $\partial^{(2)}_y$ are the total derivatives of second order.

The conditions for a vector $A\frac{\partial}{\partial r} + B\frac{\partial}{\partial s} + C\frac{\partial}{\partial t}$ in the symbol of $\mathcal{R}^2_2$ at $\mathfrak{p}^2_2$ to be tangent to the correspondence are

$$\begin{aligned} AF_p + BF_q &= 0 \\ BF_p + CF_q &= 0 \\ A + By_1 &= 0 \\ B + Cy_1 &= 0 \end{aligned}$$

This linear homogeneous system has nontrivial solutions if and only if

$$y_1 = \frac{F_q}{F_p}$$

The jet $\mathfrak{q}^1_1 \in J^1_1(J^1_2\mathbb{R}^3)$ over $\mathfrak{p}^1_2$ whose coordinates are x,y,z,p,q,z_1,p_1,q_1 defines (up to a multiplicative constant) the tangent vector

$$Y_{\mathfrak{p}^1_2} = \frac{\partial}{\partial x} + y_1\frac{\partial}{\partial y} + z_1\frac{\partial}{\partial z} + p_1\frac{\partial}{\partial p} + q_1\frac{\partial}{\partial q}$$

If we replace the values of the coefficients of $Y_{\mathfrak{p}^1_2}$ by the ones obtained in the equations of the Lie correspondence we obtain that the characteristic $(1,1)$-jets for the equation $F=0$ give rise to the tangent vectors

$$Y_{\mathfrak{p}^1_2} = \partial^{(2)}_x + y_1\partial^{(2)}_y,$$

where r, s, t are determined by the equations of $\mathcal{R}_2^2$ and the condition $Y_{\mathfrak{p}_2^1} \in T_{\mathfrak{p}_2^1}\mathcal{R}_2^1$. We have

$$r = \frac{sF_q - pF_z}{F_p}, \qquad t = \frac{sF_p - F_y - qF_z}{F_q}$$

and D must be proportional to the well known characteristic field

$$F_p\frac{\partial}{\partial x} + F_q\frac{\partial}{\partial y} + (pF_p + qF_q)\frac{\partial}{\partial z} - (F_x + pF_z)\frac{\partial}{\partial p} - (F_y + qF_z)\frac{\partial}{\partial q}$$

4.3. Monge characteristics of an equation of second order. In the same notations of the previous example, let $\alpha, \beta, \gamma, \delta$ be the coordinates in J_2^3M which represent the derivatives of third order, and let r_1, s_1, t_1 the coordinates in $J_1^1(J_2^2M)$ which stand for the derivatives of r, s, t with respect to x.

Let us consider now a single second order PDE $\mathcal{R}_2^2$ defined by

$$F(x, y, z, p, q, r, s, t) = 0.$$

In the same notations as before, the equations of $\mathcal{R}_2^3$ are

$$F = 0, \qquad \partial_x^{(3)}F = 0, \qquad \partial_y^{(3)}F = 0$$

and the Lie correspondence is given by these equations and

$$\begin{aligned} z_1 &= p + qy_1 \\ p_1 &= r + sy_1 \\ q_1 &= s + ty_1 \\ r_1 &= \alpha + \beta y_1 \\ s_1 &= \beta + \gamma y_1 \\ t_1 &= \gamma + \delta y_1 \end{aligned}$$

If there is a couple $(\mathfrak{p}_2^3, \mathfrak{q}_1^1)$ such that $\mathfrak{q}_1^1$ is characteristic, then we have

$$F_r - y_1F_s + y_1^2F_t = 0,$$

which is the condition for the 1-form $\omega = dy - y_1dx$ to be characteristic.

For each solution y_1 of the former equation the corresponding $\mathfrak{q}_1^1$ gives the vector

$$\partial_x^{(3)} + y_1\partial_y^{(3)},$$

where $\alpha, \beta, \gamma, \delta$ are determined by the equations of $\mathcal{R}_2^3$. This gives, for each y_1, a family of Monge characteristic vector fields which depends on an arbitrary function.

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References

[1] R. J. Alonso Blanco, ***Jet manifold associated to a Weil bundle***, Arch. Math. (Brno) **36** (2000), 195-199.

[2] ———, ***A transformation of PDE systems related to Drach theorem***, J. Math. Anal. Appl. **288** (2003), 530-539.

[3] R. J. Alonso Blanco and J. Muñoz, ***The contact system for the A-jet manifolds***, Arch. Math. (Brno) **40** (2004), 233-248.

[4] R. L. Bryant, P. A. Griffiths, ***Characteristic cohomology of differential systems. I. General theory***, J. Amer. Math. Soc. **8** (1995), no. 3, 507–596.

[5] R. Gardner, ***A differential geometric generalization of characteristics***, Comm. Pure Appl. Math. **22** (1969), 597–626.

[6] E. Goursat, *Leçons sur l'integration des équations aux dérivées partialles du second ordre à deux variables indépendantes,* Tom. I, Hermann, Paris, 1896.

[7] S. Jiménez, J. Muñoz, and J. Rodríguez, *On the reduction of some systems of partial differential equations to first order systems with only one unknown function*, in: Proceedings of the VIII Conference on Diff. Geom. and its Appl. (Opava 2001), Silesian Univ., Opava, 2002, 187-196.

[8] ______, ***Correspondences between jet spaces and PDE systems***, Journal of Lie Theory **15** (2005), 197-218.

[9] K. Kakié, ***Algebraic structures of characteristics in involutive systems of non-linear partial differential equations***, Publ. Res. Inst. Math. Sci. **13** (1977/78), no. 1, 107–158.

[10] I. Kolář, P. W. Michor, and J. Slovák, *Natural operations in differential geometry*, Springer-Verlag, New York, 1993.

[11] I.S. Krasil'shik, V. V. Lychagin, and A. M. Vinogradov, *Geometry of jet spaces and nonlinear partial differential equations*, Gordon and Breach, New York, 1986.

[12] M. Kuranishi, *Lectures on involutive systems*, Publicações Soc. Math. São Paulo, 1969.

[13] S. Lie, *Theorie der Transformationsgruppen*, Chelsea Publishing, New York, 1970.

[14] ______, ***Zur allgemeine Theorie der partiellen Differentialgleichungen beliebiger Ordnung***, Ges. Abh., Bd IV, 320-389.

[15] J. Muñoz, F. J. Muriel, and J. Rodríguez, ***Weil bundles and jet spaces***, Czech. Math. J. **50** (125) (2000), 721-748.

[16] ______, ***Integrability of Lie equations and Pseudogroups***, J. Math. Anal. Appl. **252** (2000),32-49.

[17] ______, ***A remark on Goldschmidt's criterion on formal integrability***, J. Math. Anal. Appl. **254** (2000),275-290.

[18] ______, ***On the finiteness of differential invariants***, J. Math. Anal. Appl. **284** (2003), 266-282.

[19] O. Stormark, *Lie's Structural Approach to PDE Systems*, Cambridge University Press, Cambridge, 2000.

[20] A. Weil, *Théorie des points proches sur les variétés différentiables*, Colloque de Géometrie Différentielle, C.N.R.S. (1953), 111–117.

Departamento de Matemáticas
Univ. de Salamanca.
Plaza de la Merced 1-4
E-37008 Salamanca (Spain)

e-mails: `ricardo@usal.es, sjv@usal.es, jrl@usal.es`