VÝSLEDKY

I. L'HOSPITALOVO PRAVIDLO A TAYLORŮV POLYNOM

1. a) 2 b) -2 c) $-\frac{1}{3}$ d) $\frac{1}{3}$ e) $-\frac{e}{2}$ f) -2 g) $\frac{1}{2}$ h) $e^{1/6}$ i) $a^a(\ln a - 1)$

2. a) $x + \frac{1}{3}x^3$, b) $1 - \frac{1}{2}x^2 + \frac{5}{24}x^4$, c) $x - \frac{1}{3}x^3 + \frac{1}{10}x^5$, d) $\frac{x^2}{2}$, e) $1 + 2x + 2x^2 - 2x^4$
f) $1 + 2x + x^2 - \frac{2}{3}x^3 - \frac{5}{6}x^4 - \frac{1}{15}x^5$, g) $1 - \frac{1}{2}x + \frac{1}{12}x^2 - \frac{1}{720}x^4$, h) $-\frac{x^2}{2} - \frac{1}{12}x^4 - \frac{1}{45}x^6$

3. a) $\frac{1}{3840}$, b) 2^{-4}

4. 1.65

5. a) $-\frac{1}{12}$, b) $\frac{1}{3}$, c) $\frac{1}{3}$, d) -$\frac{1}{4}$, e) $\log^2 a$, f) 0, g) $\frac{1}{2}$, h) $\frac{1}{3}$, i) $\frac{1}{6}$, j) $\frac{1}{2}$, k) -6

6. a) $\frac{7}{12}$ b) $-\frac{3}{2}$

II. MOCNINNÉ ŘADY

1. a) R=1; pokud $p > 1$ pak AK pro $|z| \le 1$ a D jinak; pokud $p \in (0,1]$ pak AK pro $|z| < 1$, K pro $z = e^{i\alpha}$ kde $\alpha \in (0, 2\pi)$ a D jinak; pokud $p \le 0$, pak AK pro $|z| < 1$ a D jinak
b) R=1/3, AK pro $|z+1| < R$, K pro $z = -1 + Re^{i\alpha}$ kde $\alpha \in (0, 2\pi)$, jinak D
c) R=$\frac{1}{e}$, AK pro $|z| < R$, jinak D d) R=1, pokud $a > 1$ pak AK pro $|z| \le R$ a jinak D, pokud $a \le 1$ pak AK pro $|z| < R$ a jinak D e) R $= 1/4$, AK pro $|z| < R$, K pro $z = Re^{i\alpha}$ kde $\alpha \in (0, 2\pi)$, jinak D f) R=$\max\{a, b\}$, AK pro $|z| < R$, jinak D g) R=$\frac{1}{\max\{a,b\}}$, AK pro $|z| < R$, AK pro $|z| = R$ a $b > a$, K pro $z = Re^{i\alpha}$ kde $\alpha \in (0, 2\pi)$, jinak D h) R=1/3, AK pro $|z| < R$, K pro $z = Re^{i\alpha}$ kde $\alpha \in (0, 2\pi) \setminus \{\frac{k\pi}{4} : k = 1, \ldots 7\}$, jinak D i) R=1, AK pro $|z| \le R$, jinak D

2. a) $z^5 e^{z^4},\ x \in \mathbb{R}$ b) $\frac{z}{(z-1)^2},\ z \in (-1, 1)$ c) $\log(\frac{1-x}{x}) - \log(1-x) + 1,\ 0 \ne |x| < 1$, součet je 0 pro $x = 0$
d) $\frac{x(1+x)}{(1-x)^3},\ x \in (-1, 1)$ e) $\frac{x(1-x)}{(1+x)^3},\ x \in (-1, 1)$ f) $\frac{x(3-x)}{(1-x)^3},\ x \in (-1, 1)$
g) $2x \operatorname{arctg} x - \log(1 + x^2),\ x \in (-1, 1)$ h) $\frac{e^{x/3}}{9}(x^2 + 3x - 9) + 1,\ x \in \mathbb{R}$ i) $\frac{1}{2}(x \cos x - \sin x)$
j) $\cosh x,\ x \in \mathbb{R}$`[hint:` $\sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} = \frac{1}{2} \sum_{n=0}^{\infty} \frac{1+(-1)^n}{n!}\, x^n$`]` k) $\frac{1}{2}(\cos x + \cosh x),\ x \in \mathbb{R}$
l) $\frac{x^4(5-3x^2)}{(1-x^2)^3},\ x \in (-1, 1)$

3. a) $\sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{n!},\ x \in \mathbb{R}$ b) $\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1},\ x \in (-1, 1)$
c) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} 2^{2n-1}}{(2n)!} x^{2n},\ x \in \mathbb{R}$ d) $x + \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{n+1}}{n(n+1)}$ e) $\frac{1}{3} \sum_{n=1}^{\infty} (1 - (-2)^n) z^n,\ |x| < 1/2$
f) $\operatorname{arctg} 2 + \sum_{n=1}^{\infty} \frac{(-1)^n 2^{2n-1}}{2n-1} x^{2n-1},\ |x| < 1/2$ g) $\sum_{n=1}^{\infty} n z^{n-1},\ |z| < 1$ h) $\sum_{n=0}^{\infty} \frac{x^{2n+1}}{2n+1},\ |x| < 1$

4. Poloměr konvergence je vždy $+\infty$. a) $f(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}$, $g(z) = \sum_{n=0}^{\infty} \frac{z^{n+1}}{n!}$, $h(z) = \sum_{n=0}^{\infty} \frac{z^{n+2}}{n!}$ b) $f(z) = \sum_{n=0}^{\infty} \frac{e}{n!}(z-1)^n,\ g(z) = (z-1)f(z) + f(z) = e + \sum_{n=1}^{\infty} e\left(\frac{1}{(n-1)!} + \frac{1}{n!}\right)(z-1)^n,\ h(z) = (z-1)^2 f(z) + 2(z-1)f(z) + f(z) = e + 3e(z-1) + \sum_{n=2}^{\infty} e\left(\frac{1}{(n-2)!} + \frac{1}{(n-1)!} + \frac{1}{n!}\right)(z-1)^n$ c) $f(z) = \sum_{n=0}^{\infty} \frac{(z+8)^n}{e^8 n!}$, $g(z) = (z+8)f(z) - 8f(z) = -\frac{8}{e^8} + \sum_{n=1}^{\infty} \frac{1}{e^8}\left(\frac{1}{(n-1)!} - \frac{8}{n!}\right)(z+8)^n,\ h(z) = (z+8)g(z) - 8g(z) = \frac{64}{e^8} + \frac{48}{e^8}(z+8) + \sum_{n=2}^{\infty} \frac{1}{e^8}\left(\frac{1}{(n-2)!} - \frac{16}{(n-1)!} + \frac{64}{n!}\right)(z+8)^n$

5. Poloměr konvergence je vždy $+\infty$. a) $f(z) = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!},\ g(z) = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+2}}{(2n+1)!},\ h(z) = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+3}}{(2n+1)!}$
b) $\sin z = \sin(z - \frac{\pi}{6} + \frac{\pi}{6}) = \sin(z - \frac{\pi}{6})\cos(\frac{\pi}{6}) + \cos(z - \frac{\pi}{6})\sin(\frac{\pi}{6}) = \frac{\sqrt{3}}{2} \sum_{n=0}^{\infty} (-1)^n \frac{(z-\frac{\pi}{6})^{2n+1}}{(2n+1)!} + \frac{1}{2} \sum_{n=0}^{\infty} (-1)^n \frac{(z-\frac{\pi}{6})^{2n}}{(2n)!}$, $g(z) = (z - \frac{\pi}{6})f(z) + \frac{\pi}{6}f(z) = \ldots$, $h(z) = (z - \frac{\pi}{6})g(z) + \frac{\pi}{6}g(z) = \ldots$ (nebo take $h(z) = (z - \frac{\pi}{6})^2 f(z) + \frac{\pi}{3}(z - \frac{\pi}{6})f(z) + \frac{\pi^2}{36}f(z)$)
c) $\sin z = \sin(z-2)\cos(2) + \cos(z-2)\sin(2) = \cos 2 \sum_{n=0}^{\infty} (-1)^n \frac{(z-2)^{2n+1}}{(2n+1)!} + \sin 2 \sum_{n=0}^{\infty} (-1)^n \frac{(z-2)^{2n}}{(2n)!}$, $g(z) = (z-2)f(z) + 2f(z) = \ldots$, $h(z) = (z-2)g(z) + 2g(z) = \ldots$ (nebo take $h(z) = (z-2)^2 f(z) + 4(z-2)f(z) + 4f(z)$)

III. HLEDÁNÍ PRIMITIVNÍ FUNKCE - ÚVOD

Výsledky jsou uvedeny vždy "až na konstantu":

1. a) $\frac{x^{10}}{10}+\log|x|-5e^x-\frac{1}{2x^2}-\sin x$ na $(-\infty,0)$ a $(0,\infty)$ b) $\frac{2}{3}e^{3x}+\frac{5(5-x)^{\frac{6}{5}}}{6},\ x\in\mathbb{R}$
c) $-\frac{1}{x}-\frac{3}{2x^2}-\frac{2}{x^3}$ na $(-\infty,0)$ a $(0,\infty)$ d) $-\frac{(1-x)^{11}}{11}+\frac{(1-x)^{12}}{12}$

2. a) $\frac{1}{2}|x|x,\ x\in\mathbb{R}$
b) $F(x)=\begin{cases}\sin x+4k & x\in[-\frac{\pi}{2}+2k\pi,\frac{\pi}{2}+2k\pi],k\in\mathbb{Z}\\ -\sin x+4k+2 & x\in(\frac{\pi}{2}+2k\pi,3\frac{\pi}{2}+2k\pi),k\in\mathbb{Z}\end{cases}$ c) $\frac{1}{4}|x|x^3,\ x\in\mathbb{R}$
d) $F(x)=\begin{cases}-\frac{1}{2}\cos(2x-1) & x\geq\frac{1}{2}\\ \frac{1}{2}\cos(2x-1)-1 & x<\frac{1}{2}\end{cases}$
e) $F(x)=(-1)^k(-\cos x+\sin x)+k2\sqrt{2},\ x\in[-\frac{\pi}{4}+k\pi,-\frac{\pi}{4}+(k+1)\pi],\ k\in\mathbb{Z}$
f) $F(x)=\begin{cases}\frac{x^3}{3} & x\in(-\infty,0)\\ \frac{x^2}{2} & x\in[0,1]\\ \frac{x^3}{3}+\frac{1}{6} & x\in(1,\infty)\end{cases}$ g) $F(x)=\begin{cases}e^x-2 & x<0\\ -e^{-x} & x\geq 0\end{cases}$ h) $F(x)=\begin{cases}-(x^2+x) & x<-\frac{1}{2}\\ x^2+x+\frac{1}{2} & x\geq-\frac{1}{2}\end{cases}$

3. a) $-\log|\cos x|$ na každém z intervalů $(-\frac{\pi}{2}+k\pi,\frac{\pi}{2}+k\pi),k\in\mathbb{Z}$
b) $\log|\sin x|$ na každém z intervalů $(k\pi,(k+1)\pi),k\in\mathbb{Z}$
c) $\frac{1}{3}\operatorname{tg}x^3$ na každém z intervalů $(\sqrt[3]{-\frac{\pi}{2}+k\pi},\sqrt[3]{\frac{\pi}{2}+k\pi}),k\in\mathbb{Z}$ d) $\frac{1}{2}\operatorname{arctg}x^2,\ x\in\mathbb{R}$
e) $\log|\log x|$ na $(0,1)$ a $(1,\infty)$ f) $\sqrt{x^2+5},\ x\in\mathbb{R}$ g) $\log|\log(\log x)|$ na $(1,e)$ a (e,∞)

4. a) $\frac{4(x^2+7)}{7\sqrt[4]{x}},\ D_f=(0,\infty)$ b) $2x-\frac{12}{5}\sqrt[6]{72x^5}+\frac{3}{2}\sqrt[3]{9x^2},\ D_f=\mathbb{R}\setminus\{0\}$ c) $\frac{4^x}{\log 4}+2\frac{6^x}{\log 6}+\frac{9^x}{\log 9},\ D_f=\mathbb{R}$
d) $x-\operatorname{arctg}x,\ D_f=\mathbb{R}$ e) $-\frac{2}{5}\sqrt{2-5x},\ D_f=(-\infty,\frac{2}{5})$ f) $\frac{1}{4}\operatorname{arctg}(x^4),\ D_f=\mathbb{R}$
g) $\cos(\frac{1}{x}),\ D_f=\mathbb{R}\setminus\{0\}$ h) $\frac{x}{2}+\frac{\sin(2x)}{4},\ D_f=\mathbb{R}$ i) $\operatorname{arctg}(x^2+1),\ D_f=\mathbb{R}$ j) $\frac{1}{2}\operatorname{arctg}(\sin^2 x),\ D_f=\mathbb{R}$
k) $F(x)=\begin{cases}2\log(1+x^2)+\pi(x_2-x_1)+4\log\frac{x_1}{x_2} & x\in(-\infty,-x_2]\\ -\pi(x+x_1)-2\log(1+x^2)+4\log(x_1^2+1) & x\in[-x_2,-x_1]\\ 2\log(1+x^2) & x\in[-x_1,x_1]\\ \pi(x-x_1)-2\log(1+x^2)+4\log(x_1^2+1) & x\in[x_1,x_2]\\ 2\log(1+x^2)+\pi(x_2-x_1)+4\log\frac{x_1}{x_2} & x\in[x_2,\infty)\end{cases}$ a $\begin{matrix}x_1=\frac{4-\sqrt{16-\pi^2}}{\pi}\\ x_2=\frac{4+\sqrt{16-\pi^2}}{\pi}\end{matrix}$, na $\mathbb{R}$

5. a) $-x^3\cos x+3x^2\sin x+6x\cos x-6\sin x,\ x\in\mathbb{R}$ b) $\frac{1}{2}(e^x\sin x+e^x\cos x),\ x\in\mathbb{R}$ c) $x\log x-x,\ x\in(0,\infty)$
d) $I_n:=\int x^ne^x\,\mathrm{d}x=x^ne^x-nI_{n-1};\ I_1:=xe^x-e^x,\ x\in\mathbb{R}$
e) $\frac{1}{4}(2x^2\log x-x^2),\ x\in(0,\infty)$ f) $\frac{1}{2}e^x(x\sin x+x\cos x-\sin x),\ x\in\mathbb{R}$

6. a) $\frac{2}{\sqrt{\cos x}},\ D_f=\bigcup_{k\in\mathbb{Z}}(-\frac{\pi}{2}+2k\pi,\frac{\pi}{2}+2k\pi)$ b) $\frac{1}{8}\sqrt[3]{8x^3+27},\ D_f=\mathbb{R}\setminus\{-\frac{3}{2}\}$ c) $\frac{1}{2}\operatorname{arctg}^2x,\ D_f=\mathbb{R}$
d) $-\frac{1}{2\log\frac{2}{3}}\log|1-(\frac{2}{3})^{2x}|,\ D_f=\mathbb{R}\setminus\{0\}$ e) $x\operatorname{arctg}x-\frac{1}{2}\log(1+x^2),\ D_f=\mathbb{R}$
f) $-\frac{2x^2-1}{4}\cos(2x)+\frac{x}{2}\sin(2x),\ D_f=\mathbb{R}$ g) $\frac{2}{3}x^{3/2}\left(\log^2x-\frac{4}{3}\log x+\frac{8}{9}\right),\ D_f=(0,\infty)$
h) $-\frac{e^{-2x}}{2}(x^2+x+\frac{1}{2}),\ D_f=\mathbb{R}$ i) $-\frac{1}{x}(\log^2x+2\log x+2),\ D_f=(0,\infty)$ j) $\frac{1}{3}(x^3-1)e^{x^3},\ D_f=\mathbb{R}$
k) $2(\sqrt{x}-1)e^{\sqrt{x}},\ D_f=(0,\infty)$ l) $2(6-x)\sqrt{x}\cos\sqrt{x}-6(2-x)\sin\sqrt{x},\ D_f=(0,\infty)$

7. a) $2\arcsin\frac{x}{2}+\sin(2\arcsin\frac{x}{2})=2\arcsin\frac{x}{2}+\frac{x}{2}\sqrt{4-x^2},\ D_f=(-2,2)$
b) $\operatorname{tg}(\arcsin x)=\frac{x}{\sqrt{1-x^2}},\ D_f=(-1,1)$ c) $\frac{1}{a^2}\sin(\operatorname{arctg}\frac{x}{a})=\frac{x}{a^2\sqrt{a^2+x^2}},\ D_f=\mathbb{R}$
d) $a(\arcsin\frac{x}{a}-\cos(\arcsin\frac{x}{a}))=a\arcsin\frac{x}{a}-\sqrt{a^2-x^2},\ x\in(-a,a)$
e) $2a^2(\frac{3\arcsin\sqrt{\frac{x}{2a}}}{2}-\sin(2\arcsin\sqrt{\frac{x}{2a}})+\frac{1}{8}\sin(4\arcsin\sqrt{\frac{x}{2a}})),\ D_f=(0,2a)$

IV. HLEDÁNÍ PRIMITIVNÍ FUNKCE - POKRAČOVÁVNÍ

Výsledky jsou uvedeny vždy "až na konstantu":

1. a) $\frac{1}{6}\left(\frac{12}{5}\log|x+\frac{3}{2}|+\frac{18}{5}\log|x+\frac{2}{3}|-6\log|x+1|\right),\ D_f=\mathbb{R}\setminus\{-\frac{3}{2},-1,-\frac{2}{3}\}$

b) $5\frac{x^2}{2} - 7x + 8\log|x+1| + 2\frac{1}{x+1},\ D_f = \mathbb{R}\setminus\{-1\}$ c) $x + \log|x-1| - \log|x+1|,\ D_f = \mathbb{R}\setminus\{-1,1\}$
d) $\frac{\sqrt{2}}{8}\log(x^2+\sqrt{2}x+1) + \frac{1}{2\sqrt{2}}\operatorname{arctg}(\sqrt{2}x+1) - \frac{\sqrt{2}}{8}\log(x^2-\sqrt{2}x+1) + \frac{1}{2\sqrt{2}}\operatorname{arctg}(\sqrt{2}x-1),\ D_f = \mathbb{R}$
e) $\frac{5x+2}{3(x^2+x+1)} + \frac{2}{9}\log|x-1| - \frac{1}{9}\log(x^2+x+1) + \frac{8}{3\sqrt{3}}\operatorname{arctg}(\frac{2x+1}{\sqrt{3}}),\ D_f = \mathbb{R}\setminus\{1\}$
f) $\frac{x^3}{3} + 2x^2 + 10x + 20\log|x-1| - 15\frac{1}{x-1} - 3\frac{1}{(x-1)^2},\ D_f = \mathbb{R}\setminus\{1\}$

2. a) $2\sqrt{x} - 2\log(1+\sqrt{x}),\ D_f = (0,\infty)$ `[dá se řešit substitucí` $t = \sqrt{x}$`]`
b) $6\sqrt[6]{x+1} - 3(\sqrt[6]{x+1})^2 - 2(\sqrt[6]{x+1})^3 + \frac{3}{2}(\sqrt[6]{x+1})^4 + \frac{6}{5}(\sqrt[6]{x+1})^5 - \frac{6}{7}(\sqrt[6]{x+1})^7 + 3\log(1+(\sqrt[6]{x+1})^2) - 6\operatorname{arctg}(\sqrt[6]{x+1}),\ D_f = (-1,\infty)$ `[dá se řešit substitucí` $t = \sqrt[6]{x+1}$`]`
c) $\frac{3}{4}(\sqrt[3]{2+x})^4 - \frac{3}{2}(\sqrt[3]{2+x})^2 - \frac{3}{4}\log|\sqrt[3]{2+x}-1| + \frac{15}{8}\log((\sqrt[3]{2+x})^2 + \sqrt[3]{2+x}+2) - \frac{27}{4\sqrt{7}}\operatorname{arctg}(\frac{2(\sqrt[3]{2+x})+1}{\sqrt{7}}),\ D_f = \mathbb{R}\setminus\{-1\}$ `[dá se řešit substitucí` $t = \sqrt[3]{2+x}$`]`
d) $x - \log(1+e^x) + \frac{1}{e^x+1},\ D_f = \mathbb{R}$ e) $-\frac{x}{2} + \frac{1}{3}\log|e^x-1| + \frac{1}{6}\log(e^x+2),\ D_f = \mathbb{R}\setminus\{0\}$
f) $x - 3\log(e^{x/6}+1) - 3\log(\sqrt{e^{x/3}+1}) - 3\operatorname{arctg}(e^{x/6}),\ D_f = \mathbb{R}$
g) $\frac{1}{2}\log\left(\frac{\sqrt{x+1}+\sqrt{x-1}}{\sqrt{x+1}-\sqrt{x-1}}\right) - \frac{\sqrt{x^2-1}}{x-1+\sqrt{x+1}+2\sqrt{x^2-1}},\ D_f = (1,\infty)$
3. a) $\frac{1}{4}\log\left|\frac{1-\cos x}{1+\cos x}\right| - \frac{1}{2(\cos x+1)},\ D_f = \mathbb{R}\setminus\bigcup_{k\in\mathbb{Z}}\{k\pi, \frac{\pi}{2}+k\pi\}$ `[dá se řešit substitucí` $t = \cos x$`]`
b) $\operatorname{tg}x + \log\left|\frac{\operatorname{tg}x}{(\operatorname{tg}x+1)^2}\right|,\ D_f = \mathbb{R}\setminus\bigcup_{k\in\mathbb{Z}}\{k\pi, \frac{\pi}{2}+k\pi, 3\frac{\pi}{4}+k\pi\}$ `[dá se řešit substitucí` $t = \operatorname{tg}x$`]`
c) $\frac{3}{2}\log(\cos^2 x+1) - \log(\cos^2 x),\ D_f = \mathbb{R}\setminus\bigcup_{k\in\mathbb{Z}}\{\frac{\pi}{2}+k\pi\}$
d) $\frac{1}{3}\operatorname{tg}x + \frac{2}{3\sqrt{3}}\log\left|\frac{\sqrt{3}\operatorname{tg}x-1}{\sqrt{3}\operatorname{tg}x+1}\right|,\ D_f = \mathbb{R}\setminus\bigcup_{k\in\mathbb{Z}}\{\frac{\pi}{2}+k\pi, \frac{\pi}{6}+k\pi, -\frac{\pi}{6}+k\pi\}$
e) $\frac{1}{3}\log(\cos x+2) - \frac{1}{2}\log(\cos x+1) + \frac{1}{6}\log(1-\cos x) = \frac{1}{6}\log\left(\frac{(1-\cos x)(\cos x+2)^2}{(1+\cos x)^3}\right),\ D_f = \mathbb{R}\setminus\bigcup_{k\in\mathbb{Z}}\{k\pi\}$
f) $F(x) = \begin{cases} \operatorname{arctg}(\operatorname{tg}x) - \frac{1}{\sqrt{2}}\operatorname{arctg}(\sqrt{2}\operatorname{tg}x) + k\pi(1-1/\sqrt{2}) & x\in(-\frac{\pi}{2}+k\pi, \frac{\pi}{2}+k\pi), \quad k\in\mathbb{Z} \\ \frac{\pi}{2} - \frac{\pi}{2\sqrt{2}} + k\pi(1-1/\sqrt{2}) & x = \frac{\pi}{2}+k\pi, \quad k\in\mathbb{Z} \end{cases}$
g) $F(x) = \begin{cases} \frac{1}{6}\log\left(\frac{\operatorname{tg}^2x-\operatorname{tg}x+1}{(\operatorname{tg}x+1)^2}\right) + \frac{\sqrt{3}}{3}\operatorname{arctg}\left(\frac{2\operatorname{tg}x-1}{\sqrt{3}}\right) & x\in(-\frac{\pi}{4}+k\pi, \frac{\pi}{2}+k\pi) \\ \frac{\sqrt{3}\pi}{6} & x = \frac{\pi}{2}+k\pi \\ \frac{1}{6}\log\left(\frac{\operatorname{tg}^2x-\operatorname{tg}x+1}{(\operatorname{tg}x+1)^2}\right) + \frac{\sqrt{3}}{3}\operatorname{arctg}\left(\frac{2\operatorname{tg}x-1}{\sqrt{3}}\right) + \frac{\sqrt{3}\pi}{3} & x\in(\frac{\pi}{2}+k\pi, 3\frac{\pi}{4}+k\pi) \end{cases}$ pro $k\in\mathbb{Z}$
4. a) $F(x) = \begin{cases} \frac{1}{\sqrt{6}}\operatorname{arctg}\left(\sqrt{\frac{2}{3}}\operatorname{tg}(\frac{x}{2})\right) + k\frac{\pi\sqrt{6}}{6} & \text{pro } x\in(-\pi+2k\pi, \pi+2k\pi), k\in\mathbb{Z} \\ \frac{\pi}{2\sqrt{6}} + k\frac{\pi\sqrt{6}}{6} & \text{pro } x = \pi+2k\pi, k\in\mathbb{Z} \end{cases}$
b) $\frac{1}{2}\operatorname{arctg}(\sin^2 x),\ D_f = \mathbb{R}$
c) $F(x) = \begin{cases} \frac{3\sqrt{2}}{8}\operatorname{arctg}\left(\frac{\operatorname{tg}x}{\sqrt{2}}\right) - \frac{\operatorname{tg}x}{4(\operatorname{tg}^2x+2)} + k\pi\frac{3\sqrt{2}}{8} & \text{pro } x\in(-\frac{\pi}{2}+k\pi, \frac{\pi}{2}+k\pi), k\in\mathbb{Z} \\ \frac{\pi}{2}\frac{3\sqrt{2}}{8} + k\pi\frac{3\sqrt{2}}{8} & \text{pro } x = \frac{\pi}{2}+k\pi, k\in\mathbb{Z} \end{cases}$
d) $F(x) = \begin{cases} \frac{\sqrt{5}}{5}\operatorname{arctg}\left(\frac{3\operatorname{tg}\frac{x}{2}+1}{\sqrt{5}}\right) + k\pi\frac{\sqrt{5}}{5} & \text{pro } x\in(-\pi+2k\pi, \pi+2k\pi), k\in\mathbb{Z} \\ \frac{\pi}{2}\frac{\sqrt{5}}{5} + k\pi\frac{\sqrt{5}}{5} & \text{pro } x = \pi+2k\pi, k\in\mathbb{Z} \end{cases}$
e) $\frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\log(x+\sqrt{x^2+a^2}),\ D_f = \mathbb{R}$
f) $\frac{x}{2}\sqrt{x^2-a^2} - \operatorname{sgn}(x)\frac{a^2}{2}\log(|x|+\sqrt{x^2-a^2}),\ D_f = (-\infty,a]\cup[a,\infty)$
5. a) $\frac{x}{2}\sqrt{x^2-2} + \operatorname{sgn}(x)\log(|x|+\sqrt{x^2-2}),\ D_f = (-\infty,-\sqrt{2}]\cup[\sqrt{2},\infty)$
b) $\frac{x}{2}\sqrt{x^2+a^2} - \frac{a^2}{2}\log(x+\sqrt{x^2+a^2}),\ D_f = \mathbb{R}$
c) $\frac{2}{x-\sqrt{x^2+2x+2}} - \log(\sqrt{x^2+2x+2}-x-1),\ D_f = \mathbb{R}$ d) $2\operatorname{sgn}(x-1)\sqrt{\frac{x-2}{x+1}},\ D_f = (-\infty,1)\cup(2,\infty)$

6. a) $\frac{-2(t^2+3t+1)^2}{(2t+3)^3}$ b) $\frac{-2(t^2-3t+1)^2}{(-2t+3)^3}$ c) $x_1 = \frac{3+\sqrt{5}}{2},\ x_2 = \frac{3-\sqrt{5}}{2}$. Pak na intervalu $(-\infty,x_2)$ vede na integrál z $\frac{2t^2(x_1-x_2)^2}{(t^2-1)^3}$; na intervalu (x_1,∞) vede na integrál z $-\frac{2t^2(x_1-x_2)^2}{(t^2-1)^3}$.

7. viz. výsledky ze zkouškových písemek z roku 2005/2006
(pozor, na uvedenych strankach je spatne vysledek prikladu d. Konkretne, pri substituci $t = tg(x/2)$ je ve

vysledku spatne znamenko pred logaritmem “$2\log|t-1|$”)
http://www.karlin.mff.cuni.cz/~kalenda/pis-fsv/0506/pismiii.htm

8. a) $\frac{1}{30}\log(e^x+2)-\frac{1}{2}x+\frac{1}{6}\log|e^x-1|+\frac{3}{20}\log(e^{2x}+1)-\frac{1}{10}\operatorname{arctg}(e^x)$, $D_f=(-3/2,\infty)$

b) $F(x)=\begin{cases}\log\left(\sqrt{\frac{\operatorname{tg}^2\frac{x}{2}+1}{2\operatorname{tg}^2\frac{x}{2}-\operatorname{tg}\frac{x}{2}+1}}\right)+\operatorname{arctg}(\operatorname{tg}\frac{x}{2})-\frac{6}{\sqrt{28}}\operatorname{arctg}\left(\frac{8\operatorname{tg}\frac{x}{2}-2}{\sqrt{28}}\right)- & k\pi\left(\frac{6}{\sqrt{28}}-1\right), \\ & x\in(-\pi+2k\pi,\pi+2k\pi), k\in\mathbb{Z} \\ -\frac{1}{2}\log(2)+(-k\pi-\pi/2)\left(\frac{6}{\sqrt{28}}-1\right), & x=\pi+2k\pi, k\in\mathbb{Z}\end{cases}$

c) $\frac{1}{2}\log(|2\sqrt{x^2+x}+2x+1|)-\frac{1}{2(2\sqrt{x^2+x}+2x+1)}$, $D_f=(-\infty,-1)\cup(0,\infty)$

d) $\frac{1}{5\sqrt{2}}\operatorname{arctg}(\log(x)/\sqrt{2})+\frac{9}{10\sqrt{3}}\log\left|\frac{\log(x)-\sqrt{3}}{\log(x)+\sqrt{3}}\right|$, $D_f=(0,\infty)\setminus\{e^{\sqrt{3}},e^{-\sqrt{3}}\}$

V. NEWTONŮV INTEGRÁL

1. a) 1 b) $\frac{1-\log 2}{2}$ c) 4π d) $2-\frac{2}{e}$ e) $200\sqrt{2}$ f) $2-\frac{\pi}{2}$ g) $\frac{a^4}{16}\pi$ h) $\frac{2\pi}{\sqrt{1-\varepsilon^2}}$ i) $\frac{2}{ab}\operatorname{arctg}(\frac{a\operatorname{tg}(1)}{b})$ j) $\frac{\log 3}{2}-\frac{\sqrt{3}}{6}\pi$ k) $2\sqrt{2}\pi$

2. a) $-\log(1/2)=\log 2$ b) $\frac{\pi}{\sqrt{3}}$ c) π d) $\frac{5}{2}\sqrt{2}\pi$
e) $57(\pi-2\sqrt{3}\frac{\pi}{3})+\frac{\sqrt{3}\pi}{9}+\operatorname{arctg}(\operatorname{tg}(18\pi^2))-\frac{2}{\sqrt{3}}\operatorname{arctg}\left(\frac{2\operatorname{tg}(18\pi^2)+1}{\sqrt{3}}\right)$

VII. EXISTENCE NEWTONOVA INTEGRÁLU

1. a) existuje b) neexistuje c) existuje d) existuje e) neexistuje f) existuje g) existuje, pokud $p,q<1$ h) existuje, pokud $q<1$ a $p<\frac{1}{2}$ i) existuje j) existuje, pokud $p,q>0$ k) existuje, pokud $p>-1$

2. a) existuje, pokud $m<3$ b) existuje, pokud $k<-1$ c) existuje, pokud $\alpha<-1<\alpha+\beta$ d) existuje, pokud $\alpha\in(-1,1)$ e) existuje, pokud $\alpha+\gamma>-1$ a $\beta-\gamma>-1$ f) existuje, pokud $p>1$ nebo $p=1$ a $q>1$

4. oba integrály existují

5. a) existuje b) existuje c) existuje, pokud $\alpha\in(1,3)$ d) existuje e) existuje f) existuje g) existuje pro $a\in(0,2)$

6. a) existuje b) existuje, pokud $\alpha\in(-2,-1)$ c) existuje pro $\alpha>1$

VIII. METRICKÉ PROSTORY

1. a) ano b) ne c) ne
2. a) ano b) ano c) ne d) ne e) ano
3. ano
4. f) $\frac{1}{4}$ g) f_n ano, g_n ne
5. a) $\frac{1}{6}$ b) $a=\frac{1}{\sqrt{2}}$
6. obě tvrzení platí pokud je metrika generovaná normou, jinak ne (jako příklad lze vzít prostor s diskrétní metrikou)

10. a) je uzavřená, má prázdný vnitřek, $\partial\mathbb{N}=\overline{\mathbb{N}}=\mathbb{N}$ b) $\operatorname{Int}\mathbb{Q}=\emptyset$, $\partial\mathbb{Q}=\overline{\mathbb{Q}}=\mathbb{R}$, $\mathbb{Q}$ není otevřená ani uzavřená. c) Množina není uzavřená ani otevřená, vnitřek je prázdný, hranice i uzávěr jsou $\{\frac{1}{n}: n\in\mathbb{N}\}\cup\{0\}$. d) Množina není uzavřená ani otevřená. Vnitřek je $\{[x,y]\in\mathbb{R}^2: x>0, y<0\}$, uzávěr $\{[x,y]\in\mathbb{R}^2: x\geq 0, y\leq 0\}$, hranice$\{[x,y]\in\mathbb{R}^2: x\geq 0 \;\&\; y\leq 0 \;\&\; (x=0\vee y=0)\}$. e) Otevřená, uzávěr $\{[x,y]: x+y\leq 0\}$, hranice $\{[x,y]: x+y=0\}$. f) Uzavřená, vnitřek $\{[x,y]: x>y\}$, hranice $\{[x,y]: x=y\}$. g) Uzavřená, prázdný vnitřek. h) Ani uzavřená ani otevřená, vnitřek prázdný, hranice i uzávěr $\{[x,y,z]\in\mathbb{R}^3: x\geq 0, y\geq 0, x+y=2, z\leq 0\}$. i) Uzavřená, prázdný vnitřek. j) Otevřená, uzávěr

$\{f \in \mathcal{C}[0,1] :\ f(\frac{1}{2}) \in [0,2]\}$, hranice $\{f \in \mathcal{C}[0,1] :\ f(\frac{1}{2}) \in \{0,2\}\}$ k) Uzavřená, prázdný vnitřek.

11. a) platí $\overline{A \cap B} \subset \overline{A} \cap \overline{B}$ ale naopak ne (třeba pro $A = \mathbb{Q}$, $B = \mathbb{R} \setminus \mathbb{Q}$ v $\mathbb{R}$) b) platí $\operatorname{Int}(A \setminus B) \subset \operatorname{Int}(A) \setminus \operatorname{Int}(B)$ ale naopak ne (třeba pro $A = [1,4]$, $B = [2,3]$ v $\mathbb{R}$) c) rovnost platí pro A otevřenou

12. pokud není A uzavřená, ekvivalence neplatí (například pro $x = 0$ a $A = (0,1)$ v $\mathbb{R}$)

IX. FUNKCE VÍCE PROMĚNNÝCH - LIMITY, DERIVACE

1. a) ano, $f(0,0) = 2$ b) ano, $f(0,0) = 0$ c) ano, $f(0,0) = 0$ d) ne e) ano, $f(x,x) = 3x^2$ f) ano, $f(0,0) = 0$ g) ne h) ne i) ano, $f(0,0) = 0$ j) ano, $f(0,0) = 1$ k) ano, $f(0,0) = 1$

2. a)-v), b)-i), c)-vi), d)-ii), e)-iii), f)-iv)

3. a) $D_f = \{[x,y] :\ x^2 + y^2 > 1\}$, vrstevnice jsou kružnice. b) $D_f = \{[x,y] :\ 1 \leq x^2 + y^2 \leq 4\}$, vrstevnice jsou dvojice kružnic, v jednom případě kružnice. c) $D_f = \{[x,y] :\ -x^2 - 1 \leq y \leq -x^2 + 1\}$, vrstevnice jsou dvojice parabol, v jednom případě parabola. d) $D_f = \{[x,y] :\ 2k\pi \leq x^2 + y^2 \leq (2k+1)\pi$ pro nějaké $k = 0,1,2,\dots\}$, vrstevnice jsou posloupnosti kružnic.

4. a) $\frac{\partial f}{\partial x} = mx^{m-1}y^n$, $\frac{\partial f}{\partial y} = nx^m y^{n-1}$ pro $(x,y) \in \mathbb{R}^2$. b) $\frac{\partial f}{\partial x} = ye^{xy}$, $\frac{\partial f}{\partial y} = xe^{xy}$ pro $(x,y) \in \mathbb{R}^2$. c) $\frac{\partial f}{\partial x} = y+z$, $\frac{\partial f}{\partial y} = x+y$, $\frac{\partial f}{\partial z} = x+y$ pro $(x,y,z) \in \mathbb{R}^3$. d) $\frac{\partial f}{\partial x}(x,y) = |y| \operatorname{sgn} x$ pro $x \neq 0$. $\frac{\partial f}{\partial y}(x,y) = |x| \operatorname{sgn} y$ pro $y \neq 0$. $\frac{\partial f}{\partial x}(0,0) = \frac{\partial f}{\partial y}(0,0) = 0$. $\frac{\partial f}{\partial x}(0,y)$ pro $y \neq 0$ a $\frac{\partial f}{\partial y}(x,0)$ pro $x \neq 0$ neexistují. e) $\frac{\partial f}{\partial x}(x,y) = -\operatorname{sgn}(y + \cos x) \cdot \sin x$, $\frac{\partial f}{\partial y}(x,y) = \operatorname{sgn}(y + \cos x)$, pokud $y \neq -\cos x$. $\frac{\partial f}{\partial y}(x, -\cos x)$ neexistuje pro $x \in \mathbb{R}$. $\frac{\partial f}{\partial x}(k\pi, (-1)^{k+1}) = 0$ pro $k \in \mathbb{Z}$. $\frac{\partial f}{\partial x}(x, -\cos x)$ neexistuje pro $x \neq k\pi$. f) $\frac{\partial f}{\partial x}(x,y) = \cos x \operatorname{sgn}(\sin x - \sin y)$, $\frac{\partial f}{\partial y}(x,y) = -\cos y \operatorname{sgn}(\sin x - \sin y)$, pokud $\sin x \neq \sin y$. $\frac{\partial f}{\partial x}(\frac{\pi}{2} + k\pi, \frac{\pi}{2} + l\pi) = \frac{\partial f}{\partial y}(\frac{\pi}{2} + k\pi, \frac{\pi}{2} + l\pi) = 0$. V ostatních bodech parciální derivace neexistují. g) $\frac{\partial f}{\partial x}(x,y) = -\cos x \operatorname{sgn}(\cos y - \sin x)$, $\frac{\partial f}{\partial y}(x,y) = -\sin y \operatorname{sgn}(\cos y - \sin x)$, pokud $\sin x \neq \cos x$. $\frac{\partial f}{\partial x}(\frac{\pi}{2} + k\pi, (k+2l)\pi) = \frac{\partial f}{\partial y}(\frac{\pi}{2} + k\pi + 2l\pi, k\pi) = 0$. V ostatních bodech parciální derivace neexistují. h) Pokud $x, y > 0$ nebo $x, y < 0$, pak $\frac{\partial f}{\partial x} = \frac{z}{y} \cdot \left(\frac{x}{y}\right)^{z-1}$; $\frac{\partial f}{\partial y} = -\frac{zx}{y^2} \cdot \left(\frac{x}{y}\right)^{z-1}$; $\frac{\partial f}{\partial z} = \left(\frac{x}{y}\right)^{z} \cdot \log \frac{x}{y}$. i) Pokud $x > 0$ a $z \neq 0$, pak $\frac{\partial f}{\partial x} = \frac{y}{z} \cdot x^{\frac{y}{z}-1}$; $\frac{\partial f}{\partial y} = x^{\frac{y}{z}} \cdot \log x \cdot \frac{1}{z}$; $\frac{\partial f}{\partial z} = -x^{\frac{y}{z}} \cdot \log x \cdot \frac{y}{z^2}$. j) Pokud $x > 0$, pak $\frac{\partial f}{\partial x} = \cos(x^y) \cdot x^{y-1} \cdot y$; $\frac{\partial f}{\partial y} = \cos(x^y) \cdot x^y \cdot \log x$. k) $\frac{\partial f}{\partial x} = e^{-\frac{\pi}{x^2+3xy+3y^2}} \cdot \frac{\pi(2x+3y)}{(x^2+3xy+3y^2)^2}$, $\frac{\partial f}{\partial y} = e^{-\frac{\pi}{x^2+3xy+3y^2}} \cdot \frac{\pi(3x+6y)}{(x^2+3xy+3y^2)^2}$ pro $(x,y) \neq (0,0)$; v bodě $(0,0)$ jsou obě parciální derivace nulové. l) Pokud $x > -y^2$, pak $\frac{\partial f}{\partial x} = \frac{1}{2\sqrt{x+y^2}}$; $\frac{\partial f}{\partial y} = \frac{y}{\sqrt{x+y^2}}$. Jinak parciální derivace nemají smysl.

5. viz. výsledky zkouškových písemek zde: `http://www.karlin.mff.cuni.cz/~kalenda/edu.php?edutype=archpis`

6. a) ne b) ne c) ano d) ne e) ano f) ano

7. a) $\nabla f(x,y) = \left(\frac{5x(x^4+y^4) - (x^5-y^5)4x^3}{(x^4+y^4)^2}, \frac{-5y(x^4+y^4) - (x^5-y^5)4y^3}{(x^4+y^4)^2}\right)$ pro $(x,y) \neq (0,0)$, v bodě $(0,0)$ totální diferenciál neexistuje b) $\nabla f(x,y) = (3x^2, 0)$ pro $x > y$, $\nabla f(x,y) = (0, 3y^2)$ pro $x < y$, $\nabla f(0,0) = (0,0)$, v bodech (a,a) kde $a \neq 0$ totální diferenciál neexistuje