I. Banach algebras

1. Basic properties

Definition 1. We say that $(A, +, -, 0, \cdot_{\mathrm{s}}, \cdot)$ is *algebra* over $\mathbb{K}$, if $(A, +, -, 0, \cdot_{\mathrm{s}})$ is a vector spaceover $\mathbb{K}$, $(A, +, -, \cdot, 0)$ is a ring (that is, multiplication $\cdot$ is associative and distributive with respect to addition from left and right), and moreover it holds that $(\alpha \cdot_{\mathrm{s}} a) \cdot b = a \cdot (\alpha \cdot_{\mathrm{s}} b) = \alpha \cdot_{\mathrm{s}} (a \cdot b)$ for every $a, b \in A$ and $\alpha \in \mathbb{K}$. Algebra over $\mathbb{K}$ is said to be *commutative*, if the multiplication $\cdot$ is commutative.

Let A, B be algebras over $\mathbb{K}$. (Algebra) homomorphism $\Phi\colon A \to B$ is mapping, which is a homomorphism between the corresponding vector spaces (that is, it is linear) and moreover it is homomorphism between the corresponding rings (that is, it is multiplicative, so $\Phi(ab) = \Phi(a)\Phi(b)$).

Φ is (algebraic) isomorphism of algebras A and B, if Φ is bijection.

Proposition 2. *Let A be algebra over $\mathbb{K}$. Put $A_{\mathrm{e}} = A \times \mathbb{K}$ and define vector operations on A_{e} in the usual way (that is, coordinate-wise) a moreover multiplication of elements from A_{e} are given by the formula*

$$(a, \alpha)(b, \beta) = (ab + \alpha b + \beta a, \alpha\beta) \quad \text{for } a, b \in A,\ \alpha, \beta \in \mathbb{K}.$$

Then A_{e} is algebra with unit $(0, 1)$ and A may be identified with its subalgebra $A \times \{0\}$. If A is commutative, then A_{e} is also commutative.

Definition 3. Tuple $(A, \|\cdot\|)$ is called *norm algebra*, if A is algebra, $(A, \|\cdot\|)$ is normed linear space, and for every $a, b \in A$ we have $\|ab\| \leqslant \|a\|\|b\|$. If $(A, \|\cdot\|)$ is a Banach space, then $(A, \|\cdot\|)$ is called *Banach algebra*.

Example 4. Examples of Banach algebras:

- commutative with unit: $\ell_\infty(I)$, $C_b(T)$, $C(K)$, $(\ell_1(\mathbb{Z}), *)$;
- commutative without unit: $C_0(T)$, $(L_1(\mathbb{R}^d), *)$;
- noncommutative with unit: $\mathcal{L}(X)$ (in particular M_n, $n \geqslant 2$);
- noncommutative without unit: $\mathcal{K}(X)$.

Proposition 5. *Let $(A, \|\cdot\|)$ be normed algebra. Multiplication of elements from A is lipschitz on bounded sets (and therefore continuous) as a mapping from $A \times A$ to A.*

Proposition 6. *Let $(A, \|\cdot\|)$ be a Banach algebra. If we define on A_{e} norm by the formula $\|(a, \alpha)\|_{A_{\mathrm{e}}} = \|a\| + |\alpha|$ (tj. $A_{\mathrm{e}} = A \oplus_1 \mathbb{K}$), then A_{e} with this norm is a Banach algebra.*

Definition 7. Let A and B be normed algebras and $\Phi\colon A \to B$ be (algebra) homomorphism. We say that Φ is isomorphism of normed algebras A and B (or just *isomorphism*), if Φ is homeomorphism A onto B; we say Φ is isomorphism from A into B (or just *isomorphism into*), if Φ is isomorphism A onto $\operatorname{Rng} \Phi$.

Theorem 8. *Let A be Banach algebra. For any $a \in A$ we define the left translation $L_a\colon A \to A$ by the formula $L_a(x) = ax$. Then $L_a \in \mathcal{L}(A)$ and the mapping $I\colon A \to \mathcal{L}(A)$, $I(a) = L_a$ is continuous algebra homomorfism with $\|I\| \leqslant 1$. If A has a unit e, then I is isomorphism into and $I(e) = Id$. If $\|x^2\| = \|x\|^2$ for every $x \in A$ (e.g. if A is subalgebra of $\ell_\infty(\Gamma)$), then I is isometry into.*

Corollary 9. *Let $(A, \|\cdot\|)$ be nontrivial Banach algebra with a unit. Then on A there exists an equivalent norm $|||\cdot|||$ such that $(A, |||\cdot|||)$ is Banach algebra and $|||e||| = 1$.*

Recall, that in a monoid inverse elements of invertible elements are unique and that invertible elements form a group; if $x, y \in A$ are invertible, then xy is invertible and $(xy)^{-1} = y^{-1}x^{-1}$. We denote the group of invertible elements as $A^\times$.

Fact 10. *Let $(A, \cdot, e)$ be monoid and let $x_1, \dots, x_n \in A$ commute. Then $x_1 \cdots x_n \in A^\times$ if and only if $\{x_1, \dots, x_n\} \subset A^\times$.*

Lemma 11 (Neumann series)**.** *Let A be a Banach algebra with a unit.*

(a) If $x \in U_A$, then $e - x \in A^\times$ and moreover $\sum_{n=0}^\infty x^n = (e - x)^{-1}$.

(b) Let $x \in A^\times$ a let $h \in A$ be such that $\|h\| < \frac{1}{\|x^{-1}\|}$. Then $x + h \in A^\times$ and morevoer $\left\|(x + h)^{-1} - x^{-1}\right\| \leqslant \frac{\|x^{-1}\|^2\|h\|}{1 - \|x^{-1}\|\|h\|}$.

Theorem 12. *Let A be a Banach algebra with a unit. Then $A^\times$ is open subset of A and it is a topological group.*

2. Spectral theory

Definition 13. Let A be a Banach algebra with a unit and $x \in A$. For $x \in A$ we define *resolvent set* of a point x as

$$(\rho_A(x) =) \quad \rho(x) = \{\lambda \in \mathbb{K};\ \lambda e - x \in A^\times\},$$

and *spectrum* of x as

$$(\sigma_A(x) =) \quad \sigma(x) = \mathbb{K} \backslash \rho(x).$$

On $\rho(x)$ we define *resolvent* (or *resolvent mapping*) of x by the formula

$$R_x(\lambda) = (\lambda e - x)^{-1}, \quad \lambda \in \rho(x).$$

If A does not have a unit, then for $x \in A$ we define the above notions with respect to Banach algebra A_e.

Proposition 14. *Let A be a Banach algebra.*

(a) For every $x \in A$ we have $0 \in \sigma_{A_\mathrm{e}}\big((x,0)\big)$. If A does not have a unit, then $0 \in \sigma(x)$ for every $x \in A$.

(b) If A has a unit, then $\sigma_{A_\mathrm{e}}\big((x,0)\big) = \sigma_A(x) \cup \{0\}$ for every $x \in A$.

The end of the lectures of week 1

Theorem 15. *Let A be a nontrivial complex Banach algebra and $x \in A$. Then $\sigma(x) \subset B_\mathbb{C}(0, \|x\|)$ is nonempty compact set.*

Definition 16. Let Y be a Banach space over $\mathbb{K}$, $\Omega \subset \mathbb{K}$, $f\colon \Omega \to Y$ and $a \in \Omega$. If there exists the limit $\lim\limits_{x \to a} \frac{f(x)-f(a)}{x-a} \in Y$, then we say this limit is the *derivative* of the mappint f at the point a and we denote it by $f'(a)$.

Fact 17. *Let Y be a Banach space over $\mathbb{K}$, $\Omega \subset \mathbb{K}$, $f\colon \Omega \to Y$ and $a \in \Omega$. If there exists $f'(a)$, then f is continuous at a and for every $x^* \in Y^*$ we have $(x^* \circ f)'(a) = x^*(f'(a))$.*

Proposition 18. *Let A be a Banach algebra with a unit and $x \in A$.*

(a) $\rho(x)$ is open,

(b) For $|\lambda| > \|x\|$ we have $\lambda \in \rho(x)$ and $R_x(\lambda) = \sum_{n=0}^\infty \frac{x^n}{\lambda^{n+1}}$,

(c) Resolvent mapping $\lambda \mapsto R_x(\lambda)$ has derivative at every point of the set $\rho(x)$.

(d) For every $\mu, \nu \in \rho(x)$ we have $R_x(\mu)R_x(\nu) = R_x(\nu)R_x(\mu)$.

(e) For every $\mu, \nu \in \rho(x)$ we have $R_x(\mu) - R_x(\nu) = (\nu - \mu)R_x(\mu)R_x(\nu)$ (so-called resolvent identity).

Fact. *Let G be a group. Given $u, v \in G$ satisfying $uv = vu$, we have $u^{-1}v^{-1} = v^{-1}u^{-1}$, $uv^{-1} = v^{-1}u$ and $u^{-1}v = vu^{-1}$.*

Theorem 19 (Liouville theorem)**.** *Let Y be a complex Banach space and $f\colon \mathbb{C} \to Y$ be a bounded function which has derivative at each point. Then f is constant.*

Convention 20. In the remainder of this chapter (I. Banach algebras) we will consider all the Banach spaces over the field of complex numbers (if not said explicitly otherwise).

Theorem 21 (S. Mazur (1938), I. M. Gelfand (1941))**.** *Let A be a nontrivial Banach algebra with a unit. If $A^\times = A\backslash\{0\}$, then A is isomorphic to $\mathbb{C}$. If moreover $\|e\| = 1$, then A is isometrically isomorphic to $\mathbb{C}$.*

Definition 22. Let A be a Banach algebra. For $x \in A$ we define *spectral radius* of x as

$$r(x) = \sup\{|\lambda|;\ \lambda \in \sigma(x)\}.$$

Theorem 23 (Beurling-Gelfand formula)**.** *Let A be a Banach algebra and $x \in A$. Then*

$$r(x) = \inf_{n \in \mathbb{N}} \sqrt[n]{\|x^n\|} = \lim_{n \to \infty} \sqrt[n]{\|x^n\|}.$$

Lemma 24 (spectrum and polynom)**.** *Let A be a Banach algebra with a unit and $x \in A$. If $p(z) = \sum_{j=1}^n \alpha_j z^j$ is a polynom with complex coefficients, we define $p(x) = \sum_{j=1}^n \alpha_j x^j$. Then we have $\sigma(p(x)) = p(\sigma(x))$.*

Corollary 25. *If A is a Banach algebra, $x \in A$ and $\lambda \in \mathbb{C}$, $|\lambda| > r(x)$, then the series $\sum_{n=1}^\infty \frac{x^n}{\lambda^n}$ converges absolutely. If A has a unit, then $R_x(\lambda) = \sum_{n=0}^\infty \frac{x^n}{\lambda^{n+1}}$.*

Theorem 26. *Let A be a Banach algebra with a unit, B its closed subalgebra containing e and $x \in B$. Then the following holds:*

(a) If C is a component of $\rho_A(x)$, then either $C \subset \sigma_B(x)$, or $C \cap \sigma_B(x) = \varnothing$.

(b) $\partial\sigma_B(x) \subset \sigma_A(x) \subset \sigma_B(x)$.

(c) If $\mathbb{C}\backslash\sigma_A(x)$ is connected, then $\sigma_B(x) = \sigma_A(x)$.

(d) If $\sigma_B(x)$ has an empty interior, then $\sigma_B(x) = \sigma_A(x)$.

Corollary 27. *Let A be a Banach algebra, B its closed subalgebra and $x \in B$. Then (a)-(d) in Theorem 26 holds, if we replace $\sigma_A(x)$ and $\sigma_B(x)$ by $\sigma_A(x) \cup \{0\}$ and $\sigma_B(x) \cup \{0\}$, respectively.*

Remark: proof of Corollary 27 was omitted
The end of the lectures of week 2

3. Holomorphic calculus

Let X be a Banach space, $\gamma\colon [a,b] \to \mathbb{C}$ path and $f\colon \langle\gamma\rangle \to X$ continuous mapping. Integral of f along γ is defined by the formula

$$\int_\gamma f = \int_{[a,b]} \gamma'(t) f(\gamma(t))\,\mathrm{d}\lambda(t).$$

Integral along the chain $\Gamma = \gamma_1 \dotplus \cdots \dotplus \gamma_n$ in $\mathbb{C}$ from the continuous mapping $f\colon \langle\Gamma\rangle \to X$ is defined by the formula

$$\int_\Gamma f = \int_{\gamma_1} f + \cdots + \int_{\gamma_n} f.$$

Lemma 28. *Let Γ be a chain in $\mathbb{C}$, X a Banach space, $f\colon \langle\Gamma\rangle \to X$ be continuous and $x \in X$. Then $x = \int_\Gamma f$ if and only if for every $x^* \in X^*$ we have $x^*(x) = \int_\Gamma x^* \circ f$.*

If $\Omega \subset \mathbb{C}$ is open and $K \subset \Omega$ compact, we say that a cycle Γ circulates K in Ω if $\langle\Gamma\rangle \subset \Omega\backslash K$, $\operatorname{ind}_\Gamma z = 1$ for $z \in K$ and $\operatorname{ind}_\Gamma z = 0$ for $z \in \mathbb{C}\backslash\Omega$.

Definition 29. Let A be a Banach algebra with a unit and $x \in A$. If $f \in H(\Omega)$, where $\Omega \subset \mathbb{C}$ is open neighborhood of $\sigma(x)$, then we define

$$f(x) = \frac{1}{2\pi i}\int_\Gamma f R_x = \frac{1}{2\pi i}\int_\Gamma f(\alpha)(\alpha e - x)^{-1}\,\mathrm{d}\alpha,$$

where Γ is an arbitrary cycle that circulates $\sigma(x)$ in Ω.

Remark 30. Integral in the definition $f(x)$ above exists and its value does not depend on the choice of Γ.

Theorem 31 (holomorphic calculus)**.** *Let A be a Banach algebra with a unit, $x \in A$, $\Omega \subset \mathbb{C}$ be an open neighborhood of $\sigma(x)$ and $f \in H(\Omega)$. The mapping $\Phi\colon H(\Omega) \to A$, where $\Phi(g) = g(x)$ from Definition 29, has the following properties:*

(a) Φ is algebra homomorphism, for which moreover we have $\Phi(1) = e$ and $\Phi(Id) = x$.

(b) If $f_n \to f$ locally uniformly in $H(\Omega)$, then $f_n(x) \to f(x)$.

(c) $f(x) \in A^\times$ if and only if $f(\lambda) \neq 0$ for every $\lambda \in \sigma(x)$. In this case we have $f(x)^{-1} = \frac{1}{f}(x)$.

(d) $\sigma(f(x)) = f(\sigma(x))$.

(e) If $g \in H(\Omega_1)$, where Ω_1 is open neighborhood of $f(\sigma(x))$, then $(g \circ f)(x) = g(f(x))$.

(f) If $y \in A$ commutes with x, then y commutes with $f(x)$.

Morevoer, if a mapping $\Psi\colon H(\Omega) \to A$ satisfies (a) and (b), then $\Psi = \Phi$.

Remark: The proof of properties (d)-(f) was omitted.

Lemma 32. *Let (Ω, μ) be a complete measure space, A a Banach algebra and $f \in L_1(\mu, A)$. Then for every $x \in A$ and every measurable $E \subset \Omega$ we have*

$$x\left(\int_E f d\mu\right) = \int_E x f(t)\,\mathrm{d}\mu(t) \quad a \quad \left(\int_E f\,\mathrm{d}\mu\right) x = \int_E f(t) x\,\mathrm{d}\mu(t).$$

Remark: The proof of was omitted.
The end of the lectures of week 3

4. Multiplicative linear functionals

Definition 33. Let A be a Banach algebra. Homomorphism $\varphi\colon A \to \mathbb{C}$ is said to be *multiplicative linear functional* (that is φ is linear and $\varphi(xy) = \varphi(x)\varphi(y)$ for every $x, y \in A$). The set of all the nonzero multiplicative linear functionals on A is denoted by $\Delta(A)$.

Proposition 34. *Let A be a Banach algebra and φ multiplicative linear functional.*

(a) There exists a unique extension $\widetilde{\varphi} \in \Delta(A_{\mathrm{e}})$ given by the formula $\widetilde{\varphi}(x, \lambda) = \varphi(x) + \lambda$ and $\Delta(A_{\mathrm{e}}) = \{\widetilde{\varphi};\ \varphi \in \Delta(A) \cup \{0\}\}$.

(b) For every $x \in A$ we have $\varphi(x) \in \sigma(x)$ whenever $\varphi \neq 0$.

(c) $\Delta(A) \subset B_{A^*}$ (in particular, every multiplicative linear functional on A is automatically continuous).

(d) If A has a unit and $\varphi \neq 0$, then $\|\varphi\| \geqslant \frac{1}{\|e\|}$ for every $\varphi \in \Delta(A)$. In particular, if $\|e\| = 1$, then $\Delta(A) \subset S_{A^*}$.

Theorem 35. *Let A be a Banach algebra and $M = \Delta(A) \cup \{0\} \subset (B_{A^*}, w^*)$ be the set of all the multiplicative linear functionals on A. Then M is compact, $\Delta(A)$ is locally compact and if A has a unit, then $\Delta(A)$ is compact.*

The mapping $\Phi\colon M \to \Delta(A_{\mathrm{e}})$, where $\Phi(\varphi) = \widetilde{\varphi}$ is the unique extension φ on an element of $\Delta(A_{\mathrm{e}})$, is a homeomorfism.

Example 36. (a) For K compact we have $\Delta(C(K)) = \{\delta_x\colon x \in K\}$.

(b) For $n \geqslant 2$ we have $\Delta(M_n) = \varnothing$.

Definition 37. Let A be a Banach algebra. *Ideal* in A is a vector subspace $I \subset A$ such that whenever $x \in I$ and $y \in A$, then $xy \in I$ a $yx \in I$. *Maximal ideal* in A is a proper ideal in A, which is maximal with respect to the ordering of all the proper ideals in A with respect to the inclusion.

Proposition 38. *Let A be a Banach algebra with a unit.*

(a) Every proper ideal in A is contained in a maximal ideal in A.

(b) If I is proper ideal in A, then $\overline{I}$ is proper ideal. In particular, every maximal ideal in A is closed.

Proposition 39. *Let A be a Banach algebra and $I \subset A$ closed ideal. Then the quotient A/I is Banach algebra with multiplication $q(x)q(y) = q(xy)$, where $q : A \to A/I$ is the quotient mapping.*

If A is commutative, then A/I is commutative. If A has a unit, then A/I has a unit.

Theorem 40. *Let A be a commutative Banachova algebra with a unit. Then the mapping $\Phi\colon \varphi \mapsto \operatorname{Ker}\varphi$ is bijection between $\Delta(A)$ and the set of all the maximal ideals in A.*

Lemma 41. *Let A be a commutative Banach algebra with a unit and $x \in A$ not invertible. Then xA is proper ideal.*

Corollary 42. *Let A be a commutative Banach algebra with a unit and I be a proper ideal in A. Then there exists $\varphi \in \Delta(A)$ such that $\varphi\restriction_I = 0$.*

Proposition 43. *Let A, B be Banach algebras and $\Phi\colon A \to B$ an algebra isomorphism. The the mapping $\Phi^{\#} : \Delta(B) \to \Delta(A)$ defined by the formula $\Phi^{\#}(\varphi) := \varphi \circ \Phi$, $\varphi \in \Delta(B)$ is homeomorphism.*

Proposition 44. *Let L be locally compact Hausdorff topological space. Then the mapping $\delta : L \to \Delta(C_0(L))$ defined by the formula $\delta(x) = \delta_x$, $x \in L$ is homeomorphism.*

Theorem 45. *Let K, L be locally compact Hausdorff topological spaces. Then the following assertions are equivalent:*

(i) Banach algebras $C_0(K)$ and $C_0(L)$ are isometrically izomorphic.

(ii) Algebras $C_0(K)$ and $C_0(L)$ are algebraically isomorphic.

(iii) Topological spaces K and L are homeomorphic.

The end of the lectures of week 4

Definition 46. Commutative Banachova algebra A is *semi-simple*, if $\Delta(A)$ separates the points of A, that is if $\bigcap\{\operatorname{Ker}\varphi;\ \varphi \in \Delta(A)\} = \{0\}$.

Theorem 47. *Let A, B be Banach algebras. If B is commutative and semi-simple, then every homomorphism from A to B is automatically continuous.*

Corollary 48. *Let A be a commutative and semi-simple Banach algebra. Then all the norms on A, in which A is a Banach algebra, are equivalent.*

5. Gelfand transform

Definition 49. Let A be a Banach algebra. For $x \in A$ we define $\widehat{x}\colon \Delta(A) \to \mathbb{C}$ by the formula $\widehat{x}(\varphi) = \varphi(x)$, that is, $\widehat{x} = \varepsilon_x \restriction_{\Delta(A)}$. Function $\widehat{x}$ is called the *Gelfand transform* of x.

Theorem 50. *Let A be a commutative Banach algebra and $x \in A$.*

(a) If A has a unit, then $\sigma(x) = \operatorname{Rng} \widehat{x}$.

(b) If A does not have a unit, then $\sigma(x) = \operatorname{Rng} \widehat{x} \cup \{0\}$.

(c) $\|\widehat{x}\| = r(x)$.

Definition 51. Let A be a Banach algebra. The mapping $\Gamma\colon A \to C_0(\Delta(A))$, $\Gamma(x) = \widehat{x}$ is called the *Gelfand transform* of the algebra A.

Theorem 52. *Let A be a commutative Banach algebra and Γ its Gelfand transform. The the following assertions hold:*

(a) Γ is continuous homomorphism and $\|\Gamma\| \leqslant 1$.

(b) Subalgebra $\Gamma(A) \subset C_0(\Delta(A))$ separates the points of $\Delta(A)$.

(c) Γ is one-to-one if and only if $\Delta(A)$ separates the points of A, that is, if and only if A is semi-simple.

(d) Γ is isomorphism into if and only if Γ is one-to-one and $\Gamma(A) \subset C_0(\Delta(A))$ is closed if and only if there exists $K > 0$ such that $\|x^2\| \geqslant K\|x\|^2$ for every $x \in A$.

(e) Γ is isometry into if and only if $\|x^2\| = \|x\|^2$ for every $x \in A$.

II. C*-algebras

1. Basic properties

In this chapter all the Banach spaces will be over the field of complex numbers (if not said explicitly otherwise).

Definition 53. Let A be a Banach algebra.

- Mapping ${}^\star\colon A \to A$ is an *involution* if for every $x, y \in A$ and $\lambda \in \mathbb{C}$ we have:

$$(x+y)^\star = x^\star + y^\star, \qquad (\lambda x)^\star = \overline{\lambda} x^\star, \qquad (xy)^\star = y^\star x^\star, \qquad (x^\star)^\star = x.$$

- Banach algebra A with an involution is *C*-algebra* if for every $x \in A$ we have

$$\|x^\star x\| = \|x\|^2.$$

- If A is a Banach algebra with an involution, then $x \in A$ is said to be *self-adjoint* (resp. *normal*), if $x^\star = x$ (resp. $x^\star x = x x^\star$).

Proposition 54. *Let A be a Banach algebra with involution and $x \in A$. Then the following assertions hold:*

(a) If e is left or right unit in A, then e is a unit and $e^\star = e$.

(b) A is C*-algebra if and only if for every $x \in A$ we have $\|x^\star x\| \geqslant \|x\|^2$. In this case we have $\|x\| = \|x^*\|$ for every $x \in A$.

(c) Let A have a unit. Then $x \in A^\times$ if and only if $x^\star \in A^\times$. In this case $(x^\star)^{-1} = (x^{-1})^\star$.

(d) $\lambda \in \sigma(x)$ if and only if $\overline{\lambda} \in \sigma(x^\star)$.

(e) Points $x + x^\star$, $x^\star x$, $x x^\star$ and $i(x - x^\star)$ are self-adjoint.

(f) There are unique self-adjoint elements $u, v \in A$ such that $x = u + iv$. For those we then have that $x^\star = u - iv$ and x is normal if and only if $uv = vu$.

Remark: proof of item (d) was given only for the case A has a unit

Theorem 55. *Let A be a C*-algebra and $x \in A$ be normal. Then $r(x) = \|x\|$.*

Corollary 56. *Let A be an algebra with an involution. Then on A there exists at most one norm $\|\cdot\|$ such that $(A, \|\cdot\|)$ is C*-algebra.*

Proposition 57. *Let A be a Banach algebra with an involution.*

(a) A_e is Banachova algebra with involution given by the formula $(a,\alpha)^\star = (a^\star, \overline{\alpha})$ for $(a,\alpha) \in A_e$.

(b) If A is C*-algebra, then there exists a norm $|||\cdot|||$ on A_e extending the original norm on A (and equivalent to the norm from Proposition 6) such that A_e is C*-algebra.

Remark: proof of Proposition 57 was omitted

Proposition 58. *Let A be a C*-algebra and $x \in A$.*

(a) If x is self-adjoint, then $\sigma(x) \subset \mathbb{R}$.

(b) If A has a unit and x is unitary (that is, $x^* = x^{-1}$), then $\sigma(x) \subset \{\lambda \in \mathbb{C} \colon |\lambda| = 1\}$.

The end of the lectures of week 5

Definition 59. Let A and B be algebras with involution. The algebra homomorphism $\Phi \colon A \to B$ is said to be *-*homomorphism*, if it preserves the operation *, that is $\Phi(x^\star) = \Phi(x)^\star$ for every $x \in A$.

Corollary 60. *Let A be a C*-algebra. Then every multiplicative linear funkctional on A is *-homomorphism.*

Proposition 61. *Let A, B be C*-algebras and $\Phi \colon A \to B$ be *-homomorphism. Then Φ is automatically continuous and moreover $\|\Phi\| \leqslant 1$.*

Lemma 62. *Let A, B be Banach algebras and $\Phi \colon A \to B$ is algebra homomorphism. Then for every $x \in A$ we have $\sigma_B(\Phi(x)) \subset \sigma_A(x) \cup \{0\}$.*

Theorem 63 (I. M. Gelfand a M. A. Najmark (1943))**.** *Let A be a commutative C*-algebra. Then Gelfand transformation is isometric *-isomorphism A onto $C_0(\Delta(A))$.*

Corollary 64. *Let A and B be commutative C*-algebras. Then the following assertions are equivalent:*

(i) A and B are isometrically *-isomorphic.

(ii) A and B are algebraically isomorphic.

(iii) Locally compact spaces $\Delta(A)$ and $\Delta(B)$ are homeomorphic.

Definition 65. Let A be a Banach algebra and $M \subset A$. *Algebraic hull of M* is the set

$$\operatorname{alg} M = \bigcap \{B \supset M;\ B \text{ is a subalgebra of } A\}.$$

Closed algebraic hull of M is the set

$$\overline{\operatorname{alg}}\, M = \bigcap \{B \supset M;\ B \text{ is closed subalgebra of } A\}.$$

Fact 66. *Let A be a C*-algebra and let $M \subset A$ commute and be closed under involution. Then $\overline{\operatorname{alg}}\, M$ is commutative C*-subalgebra of A.*

Theorem 67. *Let A and B be C*-algebras and $h : A \to B$ be one-to-one *-homomorphis. Then h is isometry into.*

Lemma 68. *Let K, L be Hausdorff compact spaces and $\varphi : C(K) \to C(L)$ be $*$-homomorphism satisfying $\varphi(1) = 1$. Then there exists a continuous mapping $\alpha : L \to K$ such that $\varphi(f) = f \circ \alpha$ for every $f \in C(K)$. If moreover φ is one-to-one, then $\alpha(L) = K$ and so φ is isometry into.*

7. Continous calculus for normal elements of C*-algebras

Lemma 69. *Let A be C*-algebra and B its C*-subalgebra. If A and B have common unit, then $B^\times = A^\times \cap B$. Moreover, let $x \in B$. If B has a unit which is not a unit in A, then $\sigma_A(x) = \sigma_B(x) \cup \{0\}$, in all the other cases we have $\sigma_A(x) = \sigma_B(x)$.*

Let A be a C*-algebra with a unit and $x \in A$ be normal. Put $B = \overline{\operatorname{alg}}\{e, x, x^\star\}$. Then for $f \in C(\sigma_A(x))$ we define

$$f(x) = \Gamma_B^{-1}(f \circ \Gamma_B(x)). \tag{1}$$

Theorem 70 (continuous calculus)**.** *Let A be a C*-algebra with a unit, $x \in A$ be normal and $f \in C(\sigma(x))$. The mapping $\Phi \colon C(\sigma(x)) \to A$, where $\Phi(g) = g(x)$ is defined by the formula (1), has the following properties:*

(a) Φ is isometric *-isomorphism $C(\sigma(x))$ onto $B = \overline{\operatorname{alg}}\{e, x, x^\star\}$ satisfying $\Phi(1) = e$ and $\Phi(Id) = x$.

(b) If $\Psi\colon C(\sigma(x)) \to A$ is *-homomorphism, for which $\Psi(1) = e$ and $\Psi(Id) = x$, then $\Psi = \Phi$.

(c) If $g \in H(\Omega)$, where $\Omega \subset \mathbb{C}$ is open neighborhood of $\sigma(x)$, then $\Phi(g\restriction_{\sigma(x)}) = \Psi(g)$, where Ψ is the holomorphic calculus from Theorem 31.

(d) $f(x) \in A^\times$ if and only if $f(\lambda) \neq 0$ for every $\lambda \in \sigma(x)$. In this case we have $f(x)^{-1} = \frac{1}{f}(x)$.

(e) $\sigma(f(x)) = f(\sigma(x))$.

(f) If $g \in C\big(f(\sigma(x))\big)$, then $(g \circ f)(x) = g(f(x))$.

Remark: proofs of items (c) and (f) were omitted

The end of the lectures of week 6

(g) If $y \in A$ commutes with x, then y commutes also with $f(x)$.

If A does not have a unit, we provide the whole construction in A_e. If for $f \in C(\sigma(x))$ it is true that $f(0) = 0$, then $f(x) \in A$.

Theorem 71 (Bent Fuglede (1950), Calvin R. Putnam (1951))**.** *Let A be a C*-algebra, $x \in A$, and let $a, b \in A$ be normal elements satisfying $ax = xb$. Then $a^\star x = xb^\star$.*

Remark: proof of Theorem 71 was omitted

III. Operators on Hilbert spaces

1. Basic properties

In this chapter (III. Operators on Hilbert spaces) all the Banach spaces will be over the field of complex numbers (if not said explicitly otherwise).

Definition 72. Let X, Y be vector spaces over $\mathbb{C}$. Mapping $S\colon X \times X \to Y$ is said to be *sesquilinear*, if it is linear in the first coordinate and conjugate-linear in the second coordinate. In the case $Y = \mathbb{C}$, we say S is *sesquilinear form*.

Proposition 73 (polarization identity)**.** *Let X, Y be vector spaces over $\mathbb{C}$ and $S\colon X \times X \to Y$ be sesquilinear mapping. Then for every $x, y \in X$ we have*

$$S(x,y) = \frac{1}{4}\big(S(x+y,x+y) - S(x-y,x-y) + iS(x+iy,x+iy) - iS(x-iy,x-iy)\big).$$

Corollary 74. *Let H be a nontrivial Hilbert space and $T, S \in \mathcal{L}(H)$. Then $T = S$ if and only if $\langle Tx, x\rangle = \langle Sx, x\rangle$ for every $x \in H$.*

Theorem 75. *Let H be a nontrivial Hilbert space and $T \in \mathcal{L}(H)$. Pak*

(a) T is self-adjoint if and only if $\langle Tx, x\rangle \in \mathbb{R}$ for every $x \in H$.

(b) T is normal if and only if $\|Tx\| = \|T^\star x\|$ for every $x \in H$.

(c) $\langle Tx, x\rangle \geqslant 0$ for every $x \in H$ if and only if T is self-adjoint and $\sigma(T) \subset [0, \infty)$.

Definition 76. Let A be a C*-algebra and $x \in A$. We say x is *nonnegative* (we write $x \geqslant 0$) if pokud it is self-adjoint and $\sigma(x) \subset [0, +\infty)$.

Theorem 77. *Let H be a Hilbert space and let $T \in \mathcal{L}(H)$ be normaln. Then the following assertions hold:*

(a) $\operatorname{Ker} T = \operatorname{Ker} T^\star$ a $\operatorname{Ker} T = (\operatorname{Rng} T)^\perp$.

(b) $\operatorname{Rng} T$ is dense in H if and only if T is one-to-one.

(c) $\lambda \in \sigma_p(T)$ if and only if $\overline{\lambda} \in \sigma_p(T^\star)$. Eigenspace of T corresponding to the eigenvalue λ is equal to the proper eigenspace $T^\star$ corresponding to the eigenvalue $\overline{\lambda}$.

(d) If λ_1, λ_2 are two different eigenvalues of T, then $\operatorname{Ker}(\lambda_1 I - T) \perp \operatorname{Ker}(\lambda_2 I - T)$.

Theorem 78 (Hilbert-Schmidt)**.** *Let H be a Hilbert space and $T \in \mathcal{K}(H)$ be nonzero and normal. Then there exists an orthonormal basis B of the space H formed by eigenvectors of T. There are countably many vectors from B corresponding to nonzero eigenvalues of T, and if we order those in an arbitrary one-to-one sequence $\{e_n\}_{n=1}^N$, $N \in \mathbb{N} \cup \{\infty\}$, then $\{e_n\}$ is orthonormal basis of $\overline{\operatorname{Rng} T}$ and for every $x \in H$ we have*

$$Tx = \sum_{n=1}^{N} \lambda_n \langle x, e_n\rangle e_n,$$

where λ_n is the eigenvalue corresponding to the eigenvector e_n.

Remark: proof of Theorem 78 was omitted (it is verbatim the same as the one presented in the course "Úvod do funkcionální analýzy")

Theorem 79 (Schmidt). *Let H be a Hilbert space and $T \in \mathcal{L}(H)$ be nonzero compact. Then there exists $N \in \mathbb{N}_0 \cup \{\infty\}$, sequence of positive numbers $\{\lambda_n\}_{n=1}^N$ and orthonormal systems $\{u_n\}_{n=1}^N \subset H$ a $\{v_n\}_{n=1}^N \subset H$ such that for every $x \in H$ we have*

$$Tx = \sum_{n=1}^{N} \lambda_n \langle x, u_n \rangle v_n.$$

Theorem 80. *Let H be a Hilbert space and $P \in \mathcal{L}(H)$ a projection. Then the following assertions are equivalent:*

(i) P is orthogonal, that is, $\operatorname{Rng} P \perp \operatorname{Ker} P$.

(ii) $P \geqslant 0$.

(iii) P is self-adjoint.

(iv) P is normal.

Moreover, if $P, Q \in \mathcal{L}(H)$ are two orthogonal projections, then $\operatorname{Rng}(P) \perp \operatorname{Rng}(Q)$ *if and only if $PQ = 0$.*

Definition 81. Let H, K be Hilbert spaces. Operator $T \in \mathcal{L}(H, K)$ is said to be *unitary*, if $T^{-1} = T^\star$, that is, $T^\star \circ T = I_H$ and $T \circ T^\star = I_K$.

Proposition 82. *Let H, K be Hilbert spaces and $T \in \mathcal{L}(H, K)$. Consider the following conditions:*

(i) T is unitary.

(ii) T is isometry onto.

(iii) T is isometry into.

(iv) $\langle Tx, Ty \rangle = \langle x, y \rangle$ for every $x, y \in H$.

Then (i)⇔(ii)⇒(iii)⇔(iv). *Moreover, if T is onto, then all the conditions are equivalent.*

Definition 83. Let H be a Hilbert space. Operator $U \in \mathcal{L}(H)$ is said to be *partial isometry*, if there exists a closed subspace $K \subset H$ (we say it is the *initial subspace of* U) satisfying that $U|_K$ is isometry into and $U|_{K^\perp} \equiv 0$.

Theorem 84 (polar decomposition). *Let H be a Hilbert space and $T \in \mathcal{L}(H)$.*

1. *There are unique operators $P, U \in \mathcal{L}(H)$ satisfying that $P \geqslant 0$, U is partial isometry with the initial subspace $\overline{\operatorname{Rng} P}$ and $T = UP$. Moreover, then we have $P = \sqrt{T^\star T} = U^\star T$.*
2. *If T is invertible, then there are unique operators $P, U \in \mathcal{L}(H)$ satisfying that $P \geqslant 0$ is invertible, U is unitary and $T = UP$.*

The end of the lectures of week 7

2. Borel measurable calculus for normal operators

Lemma 85 (Lax-Milgram). *Let H be a Hilbert space. If S is a sesquilinear form on H satisfying $\|S\| := \sup_{x,y \in B_H} |S(x, y)| < \infty$, then there exists a unique operator $T \in \mathcal{L}(H)$ such that $S(x, y) = \langle Tx, y \rangle$ for every $x, y \in H$. Moreover, for this operator we have $\|S\| = \|T\|$.*

Definition 86. Let H be a Hilbert space, $T \in \mathcal{L}(H)$ be normal and $\Phi\colon C(\sigma(T)) \to \mathcal{L}(H)$ is continuous calculus from Theorem 70. Then for every $x, y \in H$ we denote by $\mu_{x,y}$ the unique regular borel complex measure on $\sigma(T)$ satisfying

$$\int_{\sigma(T)} f \,\mathrm{d}\mu_{x,y} = \langle \Phi(f)x, y \rangle, \quad f \in C(\sigma(T)).$$

For every $f \in \operatorname{Bor}_b(\sigma(T))$ we moreover define $\Phi(f) \in \mathcal{L}(H)$ as the (unique) operator satisfying

$$\langle \Phi(f)x, y \rangle = \int_{\sigma(T)} f \,\mathrm{d}\mu_{x,y}, \quad x, y \in H.$$

Instead of $\phi(f)$ we write also $f(T)$.

Remark 87. Let H be a Hilbert space and $T \in \mathcal{L}(H)$ be normal.

1. The mapping $H \times H \ni (x,y) \mapsto \mu_{x,y} \in M(\sigma(T))$ from Definition 86 is sesquilinear and therefore we have
$$\mu_{x,y} = \frac{1}{4}\sum_{k=0}^{3} i^k \mu_{x+i^k y, x+i^k y}, \quad x,y \in H.$$
2. For every $x \in H$ we have $\mu_{x,x} \geqslant 0$.
3. $\mathrm{Bor}_b(\sigma(T)) \subset \ell_\infty(\sigma(T))$ is C*-algebra.
4. The mapping $\Phi\colon \mathrm{Bor}_b(\sigma(T)) \to \mathcal{L}(H)$ from Definition 86 is the extension of the continuous calculus $\Phi\colon C(\sigma(T)) \to \mathcal{L}(H)$ from Theorem 70.

Theorem 88. *Let P be a metric space. Let $\Phi \supset C_{\mathrm{b}}(P)$ be a system of functions on P, which is closed under taking pointwise limits of bounded sequences of functions. Then $\Phi = \mathrm{Bor}_b(P)$.*

Remark: proof of Theorem 88 was omitted

Definition 89. Let X, Y be normed linear spaces. On the space $\mathcal{L}(X,Y)$ we define the following locally convex topologies:

- strong operator topology τ_{SOT} generated by the system of pseudonorms $\{p_x(T) = \|Tx\|;\ x \in X\}$,
- weak operator topology τ_{WOT} generated by the system of pseudonorms $\{p_{x,f}(T) = |f(Tx)|;\ x \in X, f \in Y^*\}$.

Theorem 90 (borel calculus)**.** *Let H be a Hilbert space, $T \in \mathcal{L}(H)$ be nonzero normal operator and $f \in \mathrm{Bor}_b(\sigma(T))$. The mapping $\Phi\colon \mathrm{Bor}_b(\sigma(T)) \to \mathcal{L}(H)$ from Definition 86 has the following properties:*

(a) Φ is continuous *-homomorphism and $\|\Phi\| = 1$.

(b) If $\{f_n\} \subset \mathrm{Bor}_b(\sigma(T))$ is a bounded sequence converging pointwise to f, then $\Phi(f_n) \to \Phi(f)$ in the topology τ_{SOT}.

(c) If a compact set $K \subset \mathbb{C}$ contains $\sigma(T)$ and $\Psi\colon \mathrm{Bor}_b(K) \to \mathcal{L}(H)$ is continuous *-homomorphism, for which we have $\Psi(1) = I$, $\Psi(Id) = T$ and it satisfies the property (b) with τ_{WOT} topology, then $\Psi(g) = \Phi(g\restriction_{\sigma(T)})$ for every $g \in \mathrm{Bor}_b(K)$.

(d) $f(T)$ is normal. If f is real, then $f(T)$ is self-adjoint.

(e) $\sigma(f(T)) \subset \overline{f(\sigma(T))}$.

(f) If $g \in \mathrm{Bor}_b\big(\overline{\mathrm{Rng}\, f}\big)$, then $(g \circ f)(T) = g(f(T))$.

(g) If $S \in \mathcal{L}(H)$ commutes with T, then S commutes with $f(T)$.

Remark: proofs of items (c)-(g) were omitted
The end of the lectures of week 8

3. Integral with respect to spectral measure, spectral decomposition of normal operator

Definition 91. Let H be a Hilbert space and $(X, \mathcal{A})$ a measurable space. We say that $E : \mathcal{A} \to \mathcal{L}(H)$ is *spectral measure for* $(X, \mathcal{A}, H)$, if the following conditions hold

(a) $E(A)$ is orthogonal projection for every $A \in \mathcal{A}$,

(b) $E(X) = I$ and $E(\varnothing) = 0$,

(c) Whenever $\{A_n\colon n \in \mathbb{N}\} \subset \mathcal{A}$ are pairwise disjoint, then we have
$$E(\bigcup_{n\in\mathbb{N}} A_n)x = \sum_{n=1}^{\infty} E(A_n)x, \quad x \in H.$$

Proposition 92 (basic properties of spectral measure)**.** *Let H be a Hilbert space, $(X, \mathcal{A})$ a measurable space and E spectral measure for $(X, \mathcal{A}, H)$. Then the following conditions hold.*

(a) Whenever $A, B \in \mathcal{A}$ and $A \subset B$, then $E(A) \leqslant E(B)$,

(b) Whenever $A, B \in \mathcal{A}$, then $E(A \cap B) = E(A)E(B)$,

(c) For every $x, y \in H$ the mapping $E_{x,y} : \mathcal{A} \to \mathbb{C}$ defined by the formula $E_{x,y}(A) = \langle E(A)x, y\rangle$, $A \in \mathcal{A}$ is a complex measure with total variation $\|E_{x,y}\| \leqslant \|x\|\|y\|$.

(d) The mapping $H \times H \ni (x,y) \mapsto E_{x,y}$ is sesquilinear.

(e) For every $x, y \in H$ and $A \in \mathcal{A}$ we have $|E_{x,y}(A)| \leqslant \frac{1}{2}(E_{x,x}(A) + E_{y,y}(A))$.

(f) For every $x, y \in H$ we have $E_{x+y,x+y} \leqslant 2(E_{x,x} + E_{y,y})$.

Definition 93. Let H be a Hilbert space, $(X, \mathcal{A})$ a measurable space, E a spectral measure for $(X, \mathcal{A}, H)$ and $f : X \to \mathbb{C}$ a bounded $\mathcal{A}$-measurable function. Then the *integral of f with respect to the measure E*, is (the unique) operator $T \in \mathcal{L}(H)$ satisfying

$$\langle Tx, y\rangle = \int_X f \, \mathrm{d}E_{x,y}, \quad x, y \in H.$$

This operator is denoted by the symbol $T = \int f \, \mathrm{d}E$.

The end of the lectures of week 9

Proposition 94. *Let H be a Hilbert space, $(X, \mathcal{A})$ a measurable space, E a spectral measure for $(X, \mathcal{A}, H)$ and $f : X \to \mathbb{C}$ a bounded $\mathcal{A}$-measurable function. Then for every $\varepsilon > 0$, disjoint partition $A_1, \dots, A_n \in \mathcal{A}$ of the set X satisfying $\max\{\operatorname{diam} f(A_i) \colon i = 1, \dots, n\} < \varepsilon$ and points $x_i \in A_i$, $i = 1, \dots, n$ we have*

$$\left\| \int f \, \mathrm{d}E - \sum_{i=1}^{n} f(x_i)E(A_i) \right\| < \varepsilon.$$

Definition 95. Let $(X, \mathcal{A})$ be measurable space. Then the symbol $B(X, \mathcal{A}) \subset \ell_\infty(X)$ denotes the C*-algebra of all the bounded $\mathcal{A}$-measurable functions $f : X \to \mathbb{C}$.

Theorem 96 (properties of the integral with respect to spectral measure)**.** *Let H be a Hilbert space, $(X, \mathcal{A})$ measurable space, E spectral measure for $(X, \mathcal{A}, H)$ and let $\rho : B(X\mathcal{A}) \to \mathcal{L}(H)$ be the mapping defined by the formula $\rho(f) = \int f \, \mathrm{d}E$, $f \in B(X, \mathcal{A})$. Then the following holds:*

(a) ρ is continuous *-homomorphism, $\|\rho\| = 1$ and $\rho(1) = I$.

(b) For $f \in B(X, \mathcal{A})$ the operator $\rho(f) \in \mathcal{L}(H)$ is normal, if f is real then $\rho(f)$ is self-adjoint and $f \geqslant 0$ implies $\rho(f) \geqslant 0$.

(c) If $\{f_n\} \subset B(X, \mathcal{A})$ s a bounded sequence converging pointwise to f, then $\rho(f_n) \to \rho(f)$ in the topology τ_{WOT}.

(d) For $f \in B(X, \mathcal{A})$ and $x \in H$ we have $\|\rho(f)x\| = \sqrt{\int |f|^2 \, \mathrm{d}E_{x,x}}$.

Corollary 97 (spectral decomposition of normal operator)**.** *Let H be a Hilbert space and $T \in \mathcal{L}(H)$ be normal. Then there exists a unique spectral measure E for $(\sigma(T), \operatorname{Bor}(\sigma(T)), H)$ satisfying $\int id \, \mathrm{d}E = T$. Moreover, then we have $E(A) = \Phi(\chi_A)$ for every $A \in \operatorname{Bor}(\sigma(T))$, where $\Phi : \operatorname{Bor}_b(\sigma(T)) \to \mathcal{L}(H)$ is the borel calculus from Definition 86.*

IV. Unbounded operators

1. Unbounded operators on Banach spaces

Definition 98. Let X, Y be Banach spaces, *operator from X to Y* will denote a linear mapping T, which is defined on a linear space $D(T) \subset X$ whose range $R(T)$ is a subset of Y. If $Y = X$, we say also that T is *operator in X*.

Graph of T is the set $G(T) := \{(x, Tx) \colon x \in D(T)\} \subset X \times Y$.

Finally, let T be an operator from X to Y. Then

(a) T is *densely defined*, if $D(T)$ is dense in X;

(b) T is *closed*, if $G(T) \subset X \times Y$ is closed;

(c) operator S from X to Y is *extension of operator T*, if $G(T) \subset G(S)$ (then we write $T \subset S$);

(d) if S is an operator from X to Y, then $S + T$ is operator with $D(S + T) = D(S) \cap D(T)$ defined by the formula $(S + T)x = Sx + Tx$, $x \in D(S + T)$;

(e) if S is an operator from Y to a Banach space Z, then ST is operator with $D(ST) = \{x \in D(T) \colon Tx \in D(S)\}$ defined by the formula $(ST)x = S(Tx)$, $x \in D(ST)$;

(f) for $\alpha \in \mathbb{K}$ we define the operator αT as follows: is $\alpha = 0$, then $D(\alpha T) = X$ and $\alpha T \equiv 0$; otherwise $D(\alpha T) = D(T)$ and $(\alpha T)x = \alpha(Tx)$ for $x \in D(T)$.

Remark 99. It is easy to check that for operators S, T, V we have $(S+T)+V = S+(T+V)$, $S(TV) = (ST)V$ and $(S+T)V = SV + TV$ whenever the corresponding operators are well-defined. However, in general it is not true that $V(S + T) = VS + VT$ (in general only the inclusion "$\supset$" holds and equality holds e.g. if V is defined everywhere).

Lemma 100. *Let X, Y be Banach spaces and $L \subset X \times Y$. Then L is graph of an operatorfrom X to Y, if and only if L is a subspace satisfying $\{(x, y) \in L: x = 0\} = \{(0, 0)\}$.*

Proposition 101. *Let X, Y be Banach spaces and T be an operator from X to Y.*

(a) If $D(T) = X$ and T is closed, then $T \in \mathcal{L}(X, Y)$.

(b) The following assertions are equivalent:

(i) Operator T has closed extension.
(ii) If $(x_n, Tx_n) \to (0, y)$ in $D(T) \times Y$, then $y = 0$.
(iii) The set $\overline{G(T)} \subset X \times Y$ is graph of an operator from X to Y.

(c) If T is one-to-one and closed, then T^{-1} is closed.

Definition 102. Let X, Y be Banach spaces and T be operator from X to Y. If T has closed extension, then $\overline{T}$ is its minimal closed extension, that is, operator from X to Y satisfying $G(\overline{T}) = \overline{G(T)}$.

Proposition 103. *Let X, Y, Z be Banach spaces and T be closed operator from X to Y.*

(a) If $S \in \mathcal{L}(X, Y)$, then $S + T$ is closed and $D(S + T) = D(T)$.

(b) If $S \in \mathcal{L}(Y, Z)$, then $D(ST) = D(T)$. If S is isomorphism into, then ST is closed.

(c) If $S \in \mathcal{L}(Z, X)$, then TS is closed.

The end of the lectures of week 10

Remark 104. Addition of closed densely defined operators in X does not have to admit closed extension. Composition of closed densely defined operator in X with an operator from $\mathcal{L}(X)$ does not have to admit closed extension.

Proposition 105. *Let X, Y be Banach space and T be one-to-one closed operator from X to Y. Then the following assertions are equivalent.*

(i) $\operatorname{Rng} T = Y$ *and* $T^{-1} \in \mathcal{L}(Y, X)$.

(ii) $\operatorname{Rng} T = Y$.

(iii) $\operatorname{Rng} T$ *is dense in* Y *and* $T^{-1} \in \mathcal{L}(\operatorname{Rng} T, X)$.

Definition 106. Let X be a Banach space and T a linear opeartor in X. Resolvent set of the operator T is defined as

$$\rho(T) = \{\lambda \in \mathbb{K};\ \lambda I - T \text{ has inverse which is from } \mathcal{L}(X)\},$$

resolvent (also resolvent mapping) of T is defined by the formula

$$R_T(\lambda) = (\lambda I - T)^{-1}, \quad \lambda \in \rho(T)$$

and spectrum of T as $\sigma(T) = \mathbb{K} \setminus \rho(T)$.

Theorem 107. *Let X be a Banach space and T a linear opeartor in X. The set $\rho(T)$ is open and $\sigma(T)$ is closed. Resolvent mapping R_T has derivation at each point of the set $\rho(T)$. In particular, if X is complex, then R_T is holomorphic on $\rho(T)$.*

Lemma 108. *Let X be a Banach space and T a linear opeartor in X such that $0 \notin \sigma(T)$. Then for every nonzero $\lambda \in \mathbb{K}$ we have $\lambda \in \sigma(T)$ if and only if $\frac{1}{\lambda} \in \sigma(T^{-1})$.*

Corollary 109. *Let X be a complex Banach space and T be an operator in X such that $\sigma(T) = \varnothing$. Then $T^{-1} \in \mathcal{L}(X)$ and $\sigma(T^{-1}) = \{0\}$.*

2. Unbounded operators on Hilbert spaces - basic notions

Convention 110. In the remainder of this chapter (IV. Unbounded operators) we will consider all the Banach spaces over the field of complex numbers (if not said explicitly otherwise).

Definition 111. Let H be a Hilbert space and T be densely defined operator on H. Hilbert adjoint operator for T, denoted as $T^\star$, defined on the set

$$D(T^\star) = \{y \in H;\ x \mapsto \langle Tx, y\rangle \text{ is continuous functional on } D(T)\}.$$

For every $y \in D(T^\star)$ we define $T^\star y$ as the unique point from H, which satisfies $\langle x, T^\star y\rangle = \langle Tx, y\rangle$ for every $x \in D(T)$.

Remark 112. $D(T^\star) \subset H$ is subspace and $T^\star$ an operator on H.

Proposition 113. *Let S, T be densely defined operators in a Hilbert space H.*

(a) If $S \subset T$, then $T^\star \subset S^\star$.

(b) If $S + T$ is densely defined, then $S^\star + T^\star \subset (S + T)^\star$. If moreover $S \in \mathcal{L}(H)$, then $S^\star + T^\star = (S + T)^\star$.

(c) If ST is densely defined, then $T^\star S^\star \subset (ST)^\star$. If moreover $S \in \mathcal{L}(H)$, then $T^\star S^\star = (ST)^\star$.

The end of the lectures of week 11

Proposition 114. *Let T be densely defined operator on a Hilbert space H.*

(a) $T^\star$ is closed.

(b) T has closed extension if and only if $T^\star$ is densely defined. In this case we have $\overline{T} = T^{\star\star}$.

(c) T is closed if and only if $T^\star$ is densely defined and $T = T^{\star\star}$.

Lemma 115. *Let T be densely defined operator on a Hilbert space H and $V \in \mathcal{L}(H \oplus_2 H)$ is defined by th formula $V(x, y) = (-y, x)$, $(x, y) \in H \oplus_2 H$. Then V is unitary operator and $G(T^\star) = (V(G(T)))^\perp$.*

Proposition 116. *Let T be densely defined operator on a Hilbert space H.*

(a) $\operatorname{Ker} T^\star = (R(T))^\perp$,

(b) If moreover T is closed, then $\operatorname{Ker} T = (R(T^\star))^\perp$.

Proposition 117. *If T is one-to-one densely defined operator on a Hilbert space H and $R(T)$ is dense in H, then $T^\star$ is one-to-one and $(T^\star)^{-1} = (T^{-1})^\star$.*

Remark: proof of Proposition 117 was omitted.

Definition 118. Let T be an operator on a Hilbert space. We say T is *self-adjoint*, if $T^\star = T$. We say T is *symmetric*, if $\langle Tx, y\rangle = \langle x, Ty\rangle$ for every $x, y \in D(T)$. Moreover, we say T is maximally symmetric, if it is symmetric and there does not exist a proper symmetric extension of T.

Proposition 119. *Let T be densely defined, symmetric operator on a Hilbert space H.*

(a) T has closed extension and $\overline{T}$ is symmetric.

(b) If $D(T) = H$, then $T \in \mathcal{L}(H)$ and it is self-adjoint.

(c) If $R(T)$ is dense, then T is one-to-one.

(d) If $R(T) = H$, then T is one-to-one, self-adjoint and $T^{-1} \in \mathcal{L}(H)$.

(e) If T is self-adjoint, then it is maximally symmetic. Moreover, T is then one-to-one if and only if $R(T)$ is dense and in this case T^{-1} is self-adjoint.

Remark: proof of Proposition 119 was omitted.

Theorem 120. *Let T be a self-adjoint operator in a nontrivial Hilbert space H. Then $\emptyset \neq \sigma(T) \subset \mathbb{R}$.*

Lemma 121. *Let T be a symmetric operator on a Hilbert space H and $\lambda \in \mathbb{C}\backslash\mathbb{R}$. Then $\lambda I - T$ is one-to-one and $(\lambda I - T)^{-1}$ is continuous on $R(\lambda I - T)$. Moreover $R(\lambda I - T)$ is closed if and only if T is closed.*

The end of the lectures of week 12

Corollary 122. *Let T be an operator on a Hilbert space. H. Then the following assertions are equivalent:*

(i) T is self-adjoint.

(ii) T is densely defined, symmetric and $\sigma(T) \subset \mathbb{R}$.

(iii) T is densely defined, symmetric and there exists $\lambda \in \mathbb{C}\backslash\mathbb{R}$ such that $\lambda, \overline{\lambda} \in \rho(T)$.

3. Cayley transform

Definition 123. Let T be a symmetric operator on a Hilbert space H. Cayley transform of the operator T is defined by the formula $\mathcal{C}(T) = (T - iI) \circ (T + iI)^{-1}$.

Theorem 124. *Let T be a symmetric operator in a Hilbert space H and $\mathcal{C}(T)$ its Cayley transform.*

(a) $\mathcal{C}(T)$ is linear isometry $D(\mathcal{C}(T)) = R(T + iI)$ onto $R(\mathcal{C}(T)) = R(T - iI)$.

(b) $I - \mathcal{C}(T) = 2i(T + iI)^{-1}$, and so $I - \mathcal{C}(T)$ is one-to-one and $R(I - \mathcal{C}(T)) = D(T)$. ***The end of lectures of week 13***

(c) $T = i(I + \mathcal{C}(T))(I - \mathcal{C}(T))^{-1}$.

(d) $\mathcal{C}(T)$ is closed $\Leftrightarrow$ T is closed $\Leftrightarrow$ $D(\mathcal{C}(T))$ is closed $\Leftrightarrow$ $R(\mathcal{C}(T))$ is closed.

Remark: proof of part (d) in Theorem 124 was omitted.

Theorem 125. *Let H be a Hilbert space and U isometry from $D(U)$ onto $R(U)$. Let $I - U$ be one-to-one. Then $T = i(I + U)(I - U)^{-1}$ is symmetric and $\mathcal{C}(T) = U$. Moreover, T is densely defined if and only if $R(I - U)$ is dense.*

Theorem 126. *Let H be a Hilbert space.*

(a) Let T be a symmetric operator in H and $\mathcal{C}(T)$ its Cayley transform. Then T is self-adjoint if and only if $\mathcal{C}(T)$ is unitary.

(b) Let U be a unitary operator on H such that $I - U$ is one-to-one. Then $T = i(I + U)(I - U)^{-1}$ is self-adjoint and $\mathcal{C}(T) = U$.

Definition 127. Let T be a symmetric closed operator in a Hilbert space H. Then numbers

$$n_+(T) = \dim\left(\operatorname{Rng}(T + iI)\right)^{\perp} \quad \text{and} \quad n_-(T) = \dim\left(\operatorname{Rng}(T - iI)\right)^{\perp}$$

are called deifiency indices of the operator T.

Theorem 128. *Let T be a symmetric closed densely defined operator in a separable Hilbert space H. Then the following assertions hold.*

(a) Operator T is self-adjoint if and only if $n_+(T) = n_-(T) = 0$.

(b) Operator T is maximally symmetric if and only if $\min\{n_+(T), n_-(T)\} = 0$.

(c) Operator T has self-adjoint extension if and only if $n_+(T) = n_-(T)$.

Remark: proof of part (b) in Theorem 128 was omitted.

4. Integral of unbounded function with respect to a spectral measure

Theorem 129. *Let H be a Hilbert space, $(X, \mathcal{A})$ measurable space, E spectral measure for $(X, \mathcal{A}, H)$ and $f : X \to \mathbb{C}$ be $\mathcal{A}$-measurable function. Then*

$$D = \{x \in H;\ \int_X |f|^2 \,\mathrm{d}E_{x,x} < +\infty\}.$$

is dense subspace of H and there exists a unique operator T defined on D such that

$$\langle Tx, y\rangle = \int_X f(\lambda) \,\mathrm{d}E_{x,y}(\lambda), \quad x, y \in D. \tag{2}$$

Moreover, we have

$$\|Tx\| = \sqrt{\int_X |f(\lambda)|^2 \,\mathrm{d}E_{x,x}(\lambda)}, \quad x \in D \tag{3}$$

and if f is bounded, then $T = \int f \,\mathrm{d}E$.

Definition 130. Let H be a Hilbert space, $(X, \mathcal{A})$ measurable space, E spectral measure for $(X, \mathcal{A}, H)$ and $f : X \to \mathbb{C}$ be $\mathcal{A}$-measurable function. Then the *integral of f with respect to the spectral measure E*, is (the unique) operator T in H satisfying

$$D(T) = \{x \in H \colon \int |f|^2 \,\mathrm{d}E_{x,x} < \infty\},$$

$$\langle Tx, y\rangle = \int_X f \,\mathrm{d}E_{x,y}, \quad x, y \in D(T).$$

Then we denote this operator by the symbol $T = \int f \,\mathrm{d}E$.

Theorem 131. *Let H be a Hilbert space, $(X, \mathcal{A})$ measurable space, E spectral measure for $(X, \mathcal{A}, H)$ and $f, g : X \to \mathbb{C}$ be $\mathcal{A}$-measurable functions. Then the following assertions hold.*

(a) $\int f \, \mathrm{d}E + \int g \, \mathrm{d}E \subset \int f + g \, \mathrm{d}E$.

(b) $\left(\int f \, \mathrm{d}E\right)\left(\int g \, \mathrm{d}E\right) \subset \int fg \, \mathrm{d}E$ *and* $D\left(\left(\int f \, \mathrm{d}E\right)\left(\int g \, \mathrm{d}E\right)\right) = D\left(\int g \, \mathrm{d}E\right) \cap D\left(\int fg \, \mathrm{d}E\right)$.

(c) $\left(\int f \, \mathrm{d}E\right)^\star = \int \overline{f} \, \mathrm{d}E$ *and* $\int f \, \mathrm{d}E \left(\int f \, \mathrm{d}E\right)^\star = \int |f|^2 \, \mathrm{d}E = \left(\int f \, \mathrm{d}E\right)^\star \int f \, \mathrm{d}E$. *That is,* $\int f \, \mathrm{d}E$ *is normal.*

(d) $\int f \, \mathrm{d}E$ *is closed.*

(e) $\int f \, \mathrm{d}E \in \mathcal{L}(H)$ *if and only if there exists* $A \in \mathcal{A}$ *satisfying* $E(X \backslash A) = 0$ *and* f *is bounded on* A.

Remark: proofs of parts (a)-(c) in Theorem 131 were omitted.

Theorem 132. *Let H be a Hilbert space, $(X, \mathcal{A})$ measurable space, E spectral measure for $(X, \mathcal{A}, H)$ and $f : X \to \mathbb{C}$ be $\mathcal{A}$-measurable function. Then*

$$\sigma\left(\int f \, \mathrm{d}E\right) = \operatorname{ess} \operatorname{Rng} f := \{\lambda \in \mathbb{C};\ \forall r > 0 \colon E(f^{-1}(U(\lambda, r))) \neq 0\}.$$

Morevoer, for $\lambda \in \mathbb{C}$ we have $\operatorname{Ker}(\lambda I - \int f \, \mathrm{d}E) = \operatorname{Rng}(E(f^{-1}(\{\lambda\})))$. *Thus,* $\lambda \in \sigma_{\mathrm{p}}(\int f \, \mathrm{d}E)$ *if and only if* $E(f^{-1}(\{\lambda\})) \neq 0$.

Remark: proof of Theorem 132 was omitted.
The end of the lectures of week 14

5. Spectral decomposition of self-adjoint operators

Lemma 133. *Let H be a Hilbert space, $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ be measurable spaces, E spectral measure for $(X, \mathcal{A}, H)$ and $\varphi : X \to Y$ measurable function. Then the mapping $\varphi(E) \colon \mathcal{B} \to \mathcal{L}(H)$ defined as*

$$\varphi(E)(A) = E(\varphi^{-1}(A)), \quad A \in \mathcal{B}$$

is spectral measure for $(Y, \mathcal{B}, H)$ such that for every $\mathcal{A}$-measurable $g : Y \to \mathbb{C}$ we have

$$\int g \, \mathrm{d}\varphi(E) = \int g \circ \varphi \, \mathrm{d}E.$$

Moreover, $\int \varphi \, \mathrm{d}E = \int Id \, \mathrm{d}\varphi(E)$.

Theorem 134. *Let T be self-adjoint operator in a nontrivial Hilbert space H. Then there exists unique spectral measure E for $(\mathbb{C}, \operatorname{Bor}(\mathbb{C}), H)$ such that $T = \int Id \, \mathrm{d}E$.*
For this spectral measure E we have $E(\mathbb{C} \backslash \sigma(T)) = 0$.

Remark: proof of uniqueness in Theorem 134 is not required for the oral exam.

Corollary 135. *Let T be self-adjoint operator on a Hilbert space. Then T is continuous if and only if $\sigma(T)$ is bounded.*

The end of the lectures of week 15