I. Topological vector spaces

1. Basic properties

Definition 1. Let X be a vector space over $\mathbb{K}$ and τ be topology on X. If the operations addition and multiplication by scalar are continuous as mappings $+\colon X \times X \to X$ and $\cdot\colon \mathbb{K} \times X \to X$, we say the tuple (X, τ) is *topological vector space* (TVS). Hausdorff TVS is denoted as HTVS.

System of all the neighborhoods of a point $x \in X$ is denoted by $\tau(x)$.

Definition 2. Let X be a vector space over $\mathbb{K}$ and $A \subset X$. The set A is

- *balanced*, if $\alpha A \subset A$ for every $\alpha \in \mathbb{K}$, $|\alpha| \leqslant 1$;
- *absolutly convex*, if A is convex and balanced;
- *absorbing*, if for every $x \in X$ there exists $\lambda_x > 0$ such that $tx \in A$ for every $t \in [0, \lambda_x]$.

Absolutly convex hull of the set A is defined as

$$\operatorname{aconv} A = \bigcap\{B \supset A;\ B \subset X \text{ is absolutely convex}\}.$$

Definition 3. We say that a topolocial vector space is *locally convex* (LCS), if there exists basis of neighborhoods of 0 consisting of convex sets. Hausdorff LCS is denoted as HLCS.

Proposition 4. *Let X be TVS and $U \in \tau(0)$.*

(a) U is absorbing.

(b) There exists $V \in \tau(0)$ open and balanced satisfying $V + V \subset U$.

(c) If U is convex, then there exists $V \in \tau(0)$ open and absolutely convex satisfying $V + V \subset U$;

The end of lecture 1

Theorem 5. *Let X be TVS.*

(a) X is regular (i.e. we may separate points and closed sets by open sets).

(b) The following assertions are equivalent:

(i) X is Hausdorff.
(ii) X is T_1 (i.e. points are closed sets).
(iii) $\{0\}$ is closed set.
(iv) $\{0\} = \bigcap\{U;\ U \in \tau(0)\}$.

Remark: every TVS is even completely regular (even more generally: every topological group is completely regular).

Theorem 6 (John von Neumann (1935))**.** *Let X be a vector space and $\mathcal{U}$ a system of subsets of X containing the origin 0, which is filter basis (i.e. for every $U_1, U_2 \in \mathcal{U}$ there exists $U \in \mathcal{U}$ satisfying $U \subset U_1 \cap U_2$). Let us suppose that $\mathcal{U}$ has the following properties:*

(i) For any $U \in \mathcal{U}$ there exists $V \in \mathcal{U}$ satisfying $V + V \subset U$.

(ii) Every set from $\mathcal{U}$ is absorbing and balanced.

Then there exists a unique topology τ on X such that (X, τ) is TVS and $\mathcal{U}$ is basis of neighborhoods of 0. If members of $\mathcal{U}$ are absolutely convex sets, then (X, τ) is LCS. If moreover $\bigcap \mathcal{U} = \{0\}$, then (X, τ) is Hausdorff.

2. Topologies generated by pseudonorms, Minkowski functional

Let X be a vector space, $p_1, \dots, p_n$ pseudonorms on X and $\varepsilon > 0$. Denote

$$U_{p_1,\dots,p_n,\varepsilon} = \{x \in X;\ p_1(x) < \varepsilon, \dots, p_n(x) < \varepsilon\}.$$

If X is a vector space and $\mathcal{P}$ je system of pseudonorms on X, then *topology generated by* $\mathcal{P}$ is the smallest topology τ such that for every $p \in \mathcal{P}$ the mapping $p : (X, \tau) \to [0, \infty)$ is continuous. Then system $\mathcal{S} = \{U_{p,\varepsilon};\ p \in \mathcal{P}, \varepsilon > 0\}$ forms subbasis of neighborhoods of 0, system $\mathcal{U} = \{U_{p_1,\dots,p_n,\varepsilon};\ n \in \mathbb{N}, p_1, \dots, p_n \in \mathcal{P}, \varepsilon > 0\}$ forms basis of neighborhoods of 0 and net $\{x_\gamma\}_{\gamma \in \Gamma} \subset X$ converges to $x \in X$ in τ if and only if $p(x_\gamma - x) \to 0$ for every $p \in \mathcal{P}$.

Theorem 7. *Let X be a vector space and τ be a topology on X. Then (X, τ) is LCS if and only if τ is generated by a system of pseudonorms.*

Moreover, if τ is generated by a system of pseudonorms $\mathcal{P}$, then (X, τ) is Hausdorff if and only if for every $x \in X \backslash \{0\}$ there exists $p \in \mathcal{P}$ satisfying $p(x) > 0$.

The end of lecture 2

Definition 8. Let X be a vector space and $f: X \to \mathbb{R}$. We say f is *positively homogeneous* of $f(tx) = tf(x)$ for every $t \geqslant 0$.

Definition 9. Let X be a vector space and $A \subset X$ be absorbing. *Minkowski functional* of the set A is function $\mu_A: X \to [0, +\infty)$ defined as

$$\mu_A(x) = \inf\{\lambda > 0;\ x \in \lambda A\}.$$

Theorem 10 (Basic properties of the Minkowski functional)**.** *Let X be a vector space and $A \subset X$ be absorbing. Then:*

(a) μ_A is positively homogeneous.

(b) If A is convex, then μ_A nonnegative sublinear functional.

(c) If A is absolutly convex, then μ_A is pseudonorm.

(d) If A is convex, then $\{x \in X;\ \mu_A(x) < 1\} \subset A \subset \{x \in X;\ \mu_A(x) \leqslant 1\}$.

Definition 11. Let (X, τ_X), (Y, τ_Y) be TVS and $f: X \to Y$. We say f is *uniformly continuous*, if for every $V \in \tau_Y(0)$ there exists $U \in \tau_X(0)$ such thatfor every $x, y \in X$ we have $f(x) \in f(y) + V$ whenever $x \in y + U$.

Lemma 12. *Let X be TVS and p is sublinear functional on X. Then p is je uniformly continuous if and only if it is bounded from above on some neighborhood of 0.*

Proposition 13. *Let X be TVS and $A \subset X$ is absorbing convex set. Then μ_A is continuous if and only if A is neighborhood of 0. In this case we have*

$$\operatorname{Int} A = \{x \in X;\ \mu_A(x) < 1\} \subset A \subset \{x \in X;\ \mu_A(x) \leqslant 1\} = \overline{A}.$$

Corollary 14. *Every LCS is completely regular.*

Proposition 15. *If (X, τ) is LCS and $\mathcal{V}$ is subbasis of negihborhoods of 0 consisting of absolutely convex sets, then τ is generated by a system of pseudonorms $\{\mu_V;\ V \in \mathcal{V}\}$. Moreover, τ is also generated by the system of all the continuous pseudonorms on X.*

3. Metrizability a normability

Theorem 16. *Let (X, τ) be HTVS. Then the following assertions are equivalent:*

(i) X has countable basis of neighborhoods of 0.

(ii) X is metrizable.

(iii) X is metrizable by a translation invariant pseudometric.

If X is HLCS, then it is metrizable if and only if τ is generated by a countable system of pseudonorms $\{p_n\}$ and in this case

$$\rho(x, y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \min\{p_n(x - y), 1\} \tag{1}$$

is translation invariant pseudometric on X generating τ.

The end of lecture 3

Definition 17. Let X be TVS and $A \subset X$. The set A is *bounded*, if for every $U \in \tau(0)$ there exists $t > 0$ such that $A \subset tU$.

Proposition 18. *Let X be TVS over $\mathbb{K}$ and $A \subset X$. Then the following assertions are equivalent:*

(i) The set A is bounded.

(ii) For every sequence $\{x_n\} \subset A$ and every sequence $\{\gamma_n\} \subset \mathbb{K}$, $\gamma_n \to 0$ we have $\gamma_n x_n \to 0$.

(iii) For every sequence $\{x_n\} \subset A$ we have $\frac{1}{n} x_n \to 0$.

Moreover, if X is LCS and topology on X is generated by a system of pseudonorms $\mathcal{P}$, then the conditions above are equivalent to the fact that each $p \in \mathcal{P}$ is bounded on A.

Definition 19. Let X be a TVS. We say X is *normable* if its topology is generated by a norm.

Theorem 20 (A. N. Kolmogorov (1934))**.** *Let (X, τ) be HTVS. Then X is normable if and only if there exists a bounded convex neighborhood of 0.*

Lemma 21. *Let (X, τ) be LCS, topology τ is generated by system of pseudonorms $\mathcal{P}$ and p is a pseudonorm on X. Then p is continuous if and only if there are $p, 1, \ldots, p_n \in \mathcal{P}$ and $C > 0$ satisfying $p \leqslant C \max\{p_1, \ldots, p_n\}$.*

The end of lecture 4

4. Continuous linear mappings

Theorem 22. *Let X and Y be HTVS and $T\colon X \to Y$ linear mapping. Consider the following statements:*

(i) *T is bounded on a neighborhood of* 0.

(ii) *T is continuous at* 0.

(iii) *T is continuous.*

(iv) *T is uniformly continuous.*

(v) *$T(A)$ is bounded for every bounded $A \subset X$.*

Then (i)⇒(ii)⇔(iii)⇔(iv)⇒(v). If Y is normable, then (i)-(iv) are equivalent. If X is metrizable, then (ii)-(v) are equivalent.

Lemma 23. *Let X be a metrizable TVS. If $\{x_n\} \subset X$ converges to 0, then there exists a sequence $\{\gamma_n\} \subset \mathbb{N}$ such that $\gamma_n \to +\infty$ and $\gamma_n x_n \to 0$.*

Proposition 24. *Let X and Y be HLCS, and $T\colon X \to Y$ be a linear mapping. Let $\mathcal{P}$ be system of pseudonorms generating the topology of X and $\mathcal{Q}$ be system of pseudonorms generating the topology of Y. Then T is continuous if and only if $q \circ T$ is continuous for every $q \in \mathcal{Q}$, equivalently*

$$\forall q \in \mathcal{Q}\ \exists p_1, \ldots, p_k \in \mathcal{P} \exists C > 0 \forall x \in X : q(Tx) \leqslant C \cdot \max\{p_1(x), \ldots, p_k(x)\}.$$

Theorem 25. *Let X be HTVS and $f\colon X \to \mathbb{K}$ nonzero linear form. Then the following assertions are equivalent:*

(i) *f is continuous.*

(ii) $\operatorname{Ker} f$ *is closed.*

(iii) $\overline{\operatorname{Ker} f} \neq X$.

The end of lecture 5

As usual we shall call linear forms as (linear) *functionals*.

Definition 26. Let X be TVS. By $X^{\#}$ we will denote the space of all the linear forms (functionals) on X and we will call it the *algebraic dual*. By X^* we will denote the subspace of $X^{\#}$ consisting of those linear functionals, which are continuous on X, and we will call it the *topological dual* (or only the *dual*).

Definition 27. Let X and Y be topological vector spaces and $T\colon X \to Y$ be linear. We say T is isomorphism of X onto Y (or just *isomorphism*), if T is homeomorfism of X onto Y; we say that T is isomorphism X into Y (or just *isomorphism into*), if T is isomorphism of X onto $\operatorname{Rng} T$.

5. Finite-dimensional spaces

Definition 28. Let X be TVS and $A \subset X$. Set A is said to be *totally bounded*, if for every $U \in \tau(0)$ there exists finite $F \subset A$ such that $A \subset F + U$.

Proposition 29. *Let X be TVS. Compact subsets of X are totally bounded and totally bounded sets are bounded.*

Theorem 30. *Let X be HTVS. Then the Then the following assertions are equivalent:*

(i) $\dim X < \infty$.

(ii) *There exists $n \in \mathbb{N}$ such that X is isomorphic with* $(\mathbb{K}^n, \|\cdot\|_2)$.

(iii) *There exists totally bounded neighborhood of zero in X.*

(iv) *X is metrizable and every linear map from X into a topological vector space is continuous.*

(v) *X is metrizable and every linear form on X is continuous.*

Corollary 31. *Let X be HTVS. Then every finite-dimensional subspace of X is closed in X.*

The end of lecture 6

6. Separating theorems

Theorem 32. *Let X be LCS and $A, B \subset X$ be disjoint convex sets. Then the following holds:*

(a) If A has nonempty interior, then there exists $f \in X^*\backslash\{0\}$ satisfying $\sup_A \operatorname{Re} f \leqslant \inf_B \operatorname{Re} f$.

(b) If A is closed and B compact, then there exists $f \in X^*$ satisfying $\sup_A \operatorname{Re} f < \inf_B \operatorname{Re} f$. If moreover A is absolutely convex, then $\sup_A |f| < \inf_B \operatorname{Re} f$.

Corollary 33. *Let X be LCS. Then the following holds:*

(a) If X is Hausdorff, then X^* separates the points of X.

(b) If Y is a closed subspace of X and $x \notin Y$, then there exists $f \in X^*$ satisfying $f\restriction_Y = 0$ and $f(x) = 1$.

(c) If Y is a subspace of X and $f \in Y^*$, then there exists $F \in X^*$ satisfying $F\restriction_Y = f$.

7. Fréchet spaces

Definition 34. Let X be a metrizable HTVS.

- If the topology on X is induced by a translation invariant complete metric, we say X is *F-space*.
- If X is F-space and moreover locally convex, we say X is *Fréchet*.

Lemma 35. *Let X be HLCS, whose topology is generated by pseudonorms $\{p_n : n \in \mathbb{N}\}$ satisfying $p_1 \leqslant p_2 \leqslant \ldots$. Let ρ be translation invariant metric on X defined by the formula (1). Then a sequence $(x_k)_{k\in\mathbb{N}} \in X^{\mathbb{N}}$ is cauchy in (X, ρ), if and only if $(x_k)_k$ is cauchy in the pseudonorm p_n for every $n \in \mathbb{N}$.*

The end of lecture 7

Proposition 36. *Let X be LCS and $A \subset X$ is totally bounded set. Then $\overline{\operatorname{aconv}} A$ is totally bounded. In particular, if X is Fréchet space and $A \subset X$ is compact, then $\overline{\operatorname{aconv}} A$ is compact.*

Theorem 37 (Principle of uniform boundedness)**.** *Let X be a Fréchet space, Y be LCS and $\mathcal{A}$ system of linear continuous operators from X into Y. Then the following assertions are equivalent.*

(i) For every $x \in X$ the set $\{Tx : T \in \mathcal{A}\}$ is bounded.

(ii) Operators from $\mathcal{A}$ are uniformly continuous, that is, for every $V \in \tau_Y(0)$ there exists $U \in \tau_X(0)$ satisfying $T(U) \subset V$ for every $T \in \mathcal{A}$.

Corollary 38. *Let X be a Fréchet space, Y be LCS and $(T_n)_{n\in\mathbb{N}}$ a sequence of continuous linear mappings from X into Y such that for every $x \in X$ there exists $Tx = \lim_{n\to\infty} T_n x$. Then $T : X \to Y$ is continuous linear operator.*

Theorem 39 (Open mapping theorem)**.** *Let X and Y be F-spaces and $T : X \to Y$ continuous linear and onto. Then T is open. In particular, if T is moreover one-to-one, then T is isomorphism.*

The end of lecture 8

Theorem 40 (Closed graph theorem)**.** *Let X and Y be F-spaces and $T : X \to Y$ linear mapping. Then T is continuous if and only if T has closed graph.*

8. Weak topologies and polars

Weak topologies

Definition 41. Let X be a vector space and $M \subset X^{\#}$. By $\sigma(X, M)$ we denote the locally convex topology on X generated by the system of pseudonorms $\{|f|;\ f \in M\}$.

Definition 42. Let X be a TVS.

- Topology $w = \sigma(X, X^*)$ is called the *weak topology* (also w-topology) on X.
- Topology $w^* = \sigma(X^*, \varepsilon(X))$ is called the *weak star topology* (also w^*-topology) on X^*.

Lemma 43. *Let X be a vector space and $f, f_1, \ldots, f_n$ linear forms on X. Then $f \in \operatorname{span}\{f_1, \ldots, f_n\}$ if and only if $\bigcap_{j=1}^n \operatorname{Ker} f_j \subset \operatorname{Ker} f$.*

Proposition 44. *Let X be a vector space and $M, N \subset X^{\#}$. Then $\sigma(X, M) = \sigma(X, N)$ if and only if $\operatorname{span} M = \operatorname{span} N$. In particular, $\sigma(X, M) = \sigma(X, \operatorname{span} M)$.*

Theorem 45. *Let X be a vector space and $M \subset X^{\#}$. Then $(X, \sigma(X, M))^* = \operatorname{span} M$.*

Corollary 46. *Let (X, τ) be LCS. Then*

(a) $w \subset \tau$ a $(X, w)^* = X^*$.

(b) $(X^*, w^*)^* = \varepsilon(X)$.

(c) If X is normed linear space and $f \in X^{**}$, then $f \in \varepsilon(X)$ if and only if f is w^*-continuous.

Proposition 47. *Let X be LCS and Y be subspace of X. Then on Y the topology $\sigma(Y, Y^*)$ coincides with the restriction of the topology $\sigma(X, X^*)$ on Y.*

The end of lecture 9

Theorem 48 (Mazur theorem). *Let X be LCS and $A \subset X$ be convex. Then*

(a) $\overline{A}^w = \overline{A}$.

(b) A is weakly closed if and only if it is closed.

(c) If X is metrizable and $x_n \to x$ weakly, then there are $y_n \in \operatorname{conv}\{x_j;\ j \geqslant n\}$ such that $y_n \to x$.

Theorem 49 (Mackey). *Let X be LCS and $A \subset X$. Then A is bounded if and only if it is weakly bounded.*

Theorem 50. *Let X, Y be HLCS and $T\colon X \to Y$ continuous linear mapping. Then*

(a) T is w–w continuous.

(b) Define $T^* : Y^* \to X^*$ by the formula $T^* f = f \circ T$, $f \in Y^*$. Then T^* is w^*-w^* continuous.

Polars

Definition 51. If X is LCS and $A \subset X$, we define *(absolut) polar* of the set A as

$$A^\circ = \{f \in X^*;\ |f(x)| \leqslant 1 \text{ for every } x \in A\}.$$

For a set $B \subset X^*$ we define *backwards (absolute) polar* as

$$B_\circ = \{x \in X;\ |f(x)| \leqslant 1 \text{ for every } f \in B\}.$$

Theorem 52 (Bipolar Theorem; Jean Dieudonné (1950)). *Let X be LCS.*

(a) If $A \subset X$, then $(A^\circ)_\circ = \overline{\operatorname{aconv}}^{w} A$ $(= \overline{\operatorname{aconv}} A$, if X is locally convex).

(b) If $B \subset X^*$, then $(B_\circ)^\circ = \overline{\operatorname{aconv}}^{w^*} B$.

Corollary 53. *Let X be normed linear space.*

(a) For $B \subset X^*$ we have $(B_\perp)^\perp = \overline{\operatorname{span}}^{w^*} B$.

(b) If Y is normed linear space and $T \in \mathcal{L}(X, Y)$, then $\overline{\operatorname{Rng} T^*}^{w^*} = (\operatorname{Ker} T)^\perp$.

The end of lecture 10

Theorem 54 (Herman Heine Goldstine (1938)). *If X is normed linear space, then $\overline{\varepsilon(B_X)}^{w^*} = B_{X^{**}}$.*

Theorem 55 (Banach-Alaoglu-Bourbaki). *Let X be HLCS and U neighborhood of 0 in X.*

(a) U° is w^*-compact set.

(b) If X is separable and $\{x_n\}_{n=1}^\infty$ is dense in X, then (U°, w^*) is topological space metrizable by the metric

$$\rho(f, g) = \sum_{n=1}^{\infty} \frac{1}{2^n} \min\{|(f-g)(x_n)|, 1\}.$$

Corollary 56. *Let X be normed linear space. Then (B_{X^*}, w^*) is compact. Morevoer, if X is separable, then (B_{X^*}, w^*) is moreover metrizable.*

Proposition 57. *Let X be normed linear space, X^* is separable and $\{f_n\}$ is dense in S_{X^*}. Then (B_X, w) is metrizable by the metric*

$$\rho(x,y) = \sum_{n=1}^{\infty} \frac{1}{2^n} |f_n(x-y)|.$$

Theorem 58. *If X is a Banach space, then X is reflexive if and only if (B_X, w) is compact. Morevoer, if X is separable then (B_X, w) is metrizable.*

Corollary 59. *Let X be a Banach space. Then X is reflexive if and only if weak and weak-star topologies coincide on X^*.*

The end of lecture 11

II. Distributions

1. Space of test functions

Lemma 60. *Let $\Omega \subset \mathbb{R}^d$ be open.*

(a) Let μ be borel complex (resp. signed) measure on Ω. If $\int_\Omega \varphi \,\mathrm{d}\mu = 0$ for every nonnegative $\varphi \in \mathcal{D}(\Omega, \mathbb{R})$, then $\mu = 0$.

(b) Let $f \in L_1^{\mathrm{loc}}(\Omega, \lambda)$. If $\int_\Omega f\varphi \,\mathrm{d}\lambda = 0$ for every nonnegative $\varphi \in \mathcal{D}(\Omega, \mathbb{R})$, then $f = 0$ a. e. on Ω.

(c) Let μ be borel complex (resp. signed) measure on Ω and $f \in L_1^{\mathrm{loc}}(\Omega, \lambda)$. If $\int_\Omega \varphi \,\mathrm{d}\mu = \int_\Omega f\varphi \,\mathrm{d}\lambda$ for every nonnegative $\varphi \in \mathcal{D}(\Omega, \mathbb{R})$, then $f \in L_1(\Omega, \lambda)$ and $\mu(A) = \int_A f \,\mathrm{d}\lambda$ for every borel $A \subset \Omega$.

Lemma 61. *Let $K \subset \mathbb{R}^d$ be compact and $G \subset \mathbb{R}^d$ be open, $G \supset K$. Then there are $U \subset G$ open, $U \supset K$ and $\varphi \in \mathcal{D}(G)$ such that $0 \leqslant \varphi \leqslant 1$ and $\varphi = 1$ on U.*

Definition 62. Let $K \subset \mathbb{R}^d$ be compact, then the symbol τ_K denotes the metrizable local convex topology on $\mathcal{D}(K)$ generated by the countable system of norms $\|\cdot\|_N$, $N \in \mathbb{N}_0$, where

$$\|\varphi\|_N = \max_{|\alpha| \leqslant N} \|D^\alpha \varphi\|_\infty, \qquad \varphi \in \mathcal{D}(K),\ N \in \mathbb{N}_0.$$

Symbol τ_{C^∞} denotes the topology on $C^\infty(\mathbb{R}^d)$ generated by the countable system of pseudonorms $|\cdot|_N$, $N \in \mathbb{N}_0$, where

$$|f|_N = \max_{|\alpha| \leqslant N} \|D^\alpha f|_{B(0,N)}\|_\infty, \qquad f \in C^\infty(\mathbb{R}^d),\ N \in \mathbb{N}_0.$$

The end of lecture 12

Proposition 63. *$(C^\infty(\mathbb{R}^d), \tau_{C^\infty})$ and $(\mathcal{D}(K), \tau_K)$ are Fréchet spaces for every $K \subset \mathbb{R}^d$ compact.*

Theorem 64. *Let $\Omega \subset \mathbb{R}^d$ be open and nonempty. Put*

$$\mathcal{U} = \big\{U \subset \mathcal{D}(\Omega);\ U \text{ absolutely convex, } U \cap \mathcal{D}(K) \in \tau_K(0) \text{ for every compact } K \subset \Omega\big\}.$$

Then $\mathcal{U}$ is basis of neighborhoods of 0 for a Hausdorff locally convex topology τ on $\mathcal{D}(\Omega)$, which has the following properties:

(a) For every compact $K \subset \Omega$ the space $\mathcal{D}(K)$ is closed subspace of $(\mathcal{D}(\Omega), \tau)$ and $\tau\restriction_{\mathcal{D}(K)} = \tau_K$.

(b) If $A \subset (\mathcal{D}(\Omega), \tau)$ is bounded, then there exists $K \subset \Omega$ compact such that $A \subset \mathcal{D}(K)$.

(c) Let $\{\varphi_n\}$ be a sequence in $\mathcal{D}(\Omega)$ and $\varphi \in \mathcal{D}(\Omega)$. Then $\varphi_n \to \varphi$ v τ if and only if there exists a compact $K \subset \Omega$ such that $\operatorname{supp} \varphi_n \subset K$ for every $n \in \mathbb{N}$, and for every multiindex α of length d we have $D^\alpha \varphi_n \to D^\alpha \varphi$ uniformly on $\mathbb{R}^d$.

2. Space of distributions

Definition 65. Let $\Omega \subset \mathbb{R}^d$ be open and nonempty. Then $\mathcal{D}(\Omega)^* = \big(\mathcal{D}(\Omega), \tau\big)^*$ is the *space of distributions*. We say Λ is *distribution*, if $\Lambda \in \mathcal{D}(\Omega)^*$.

Proposition 66 (characterization of distributions)**.** *Let $\Omega \subset \mathbb{R}^d$ be open nonempty, Y be HLCS and $\Lambda\colon (\mathcal{D}(\Omega), \tau) \to Y$ is linear. Then the following conditions are equivalent:*

(i) Λ is continuous.

(ii) Λ *is sequentially continuous, that is,* $\Lambda(\varphi_n) \to \Lambda(\varphi)$ *whenever* $\varphi_n \xrightarrow{\tau} \varphi$.

(iii) For every compact $K \subset \Omega$ *the restriction* $\Lambda\restriction_{\mathcal{D}(K)}$ *is continuous.*

Morevoer, if $Y = \mathbb{K}$*, then the conditions above are equivalent to the condition*

(iv) For every compact $K \subset \Omega$ *there are* $N \in \mathbb{N}_0$ *and* $C > 0$ *satisfying* $|\Lambda(\varphi)| \leqslant C\|\varphi\|_N$ *for every* $\varphi \in \mathcal{D}(K)$.

Remark: proof was given only for the case $Y = \mathbb{K}$. **The end of lecture 13**

Definition 67. Let $\Omega \subset \mathbb{R}^d$ be open nonempty and $\Lambda \in \mathcal{D}(\Omega)^*$. If there is $N \in \mathbb{N}_0$ such that for every compact $K \subset \Omega$ there exists $C \geqslant 0$ such that $|\Lambda(\varphi)| \leqslant C\|\varphi\|_N$ for any $\varphi \in \mathcal{D}(K)$, then the smallest N with this propery is called *order of the distribution* Λ. If such N does not exist, we say that the order of Λ is infinity.

Examples 68. Let $\Omega \subset \mathbb{R}^d$ be open nonempty.

(i) For $f \in L_1^{\mathrm{loc}}(\Omega, \lambda)$ we define $\Lambda_f(\varphi) = \int_\Omega f\varphi \,\mathrm{d}\lambda$, $\varphi \in \mathcal{D}(\Omega)$. Then Λ_f is distribution of order 0 and whenever $\Lambda_f = \Lambda_g$ for some $g \in L_1^{\mathrm{loc}}(\Omega, \lambda)$, then $f = g$ a.e.

(ii) Let μ be borel complex (resp. signed) measure on Ω. We define $\Lambda_\mu(\varphi) = \int_\Omega \varphi \,\mathrm{d}\mu$, $\varphi \in \mathcal{D}(\Omega)$. Then Λ_μ is distribution of order 0. Whenever $\Lambda_\mu = \Lambda_\nu$ for some borel complex (resp. signed) measure ν, then $\mu = \nu$. Whenever $\Lambda_\mu = \Lambda_f$ for some $f \in L_1^{\mathrm{loc}}(\Omega, \lambda)$, then $\mu = f \,\mathrm{d}\lambda$.

(iii) Let μ be nonnegative borel regular measure on Ω, which is finite on compact sets. We define $\Lambda_\mu(\varphi) = \int_\Omega \varphi \,\mathrm{d}\mu$ for $\varphi \in \mathcal{D}(\Omega)$. Then Λ_μ is distribution of order 0. Whenever $\Lambda_\mu = \Lambda_\nu$ for some measure ν, then $\mu = \nu$. Whenever $\Lambda_\mu = \Lambda_f$ for some $f \in L_1^{\mathrm{loc}}(\Omega, \lambda)$, then $\mu = f \,\mathrm{d}\lambda$.

(iv) Necht' $k \in \mathbb{N}$. We define $\Lambda(\varphi) = \varphi^{(k)}(0)$ for $\varphi \in \mathcal{D}(\mathbb{R})$. Then Λ is distribution of order k, which is not of order $k - 1$.

(v) We define $\Lambda(\varphi) = \sum_{n=1}^\infty \varphi^{(n)}(n)$ for $\varphi \in \mathcal{D}(\mathbb{R})$. Then Λ is distribution of order infinity.

Definition 69. Let $\Omega \subset \mathbb{R}^d$ be open nonempty and $\Lambda \in \mathcal{D}(\Omega)^*$. For multiindex α of length d we define *derivation* D^α *of the distribution* Λ as a functional on $\mathcal{D}(\Omega)$ given by the formula

$$(D^\alpha\Lambda)(\varphi) = (-1)^{|\alpha|}\Lambda(D^\alpha\varphi).$$

For a function$f \in C^\infty(\Omega)$ we define *multiplication of the function* f *and distribution* Λ as a functional on $\mathcal{D}(\Omega)$ given by the formula

$$(f\Lambda)(\varphi) = \Lambda(f\varphi).$$

Lemma 70. *Let* $k \in \mathbb{N}$*,* $f \in C^k(\mathbb{R}^d)$ *has a compact support and let* $\alpha \in \mathbb{N}_0^d$*,* $|\alpha| \leqslant k$*. Then*

$$\int_{\mathbb{R}^d} D^\alpha f\varphi \,\mathrm{d}\lambda = (-1)^{|\alpha|}\int_{\mathbb{R}^d} fD^\alpha\varphi \,\mathrm{d}\lambda$$

for every $\varphi \in \mathcal{D}(\mathbb{R}^d)$.

Proposition 71. *Let* $\Omega \subset \mathbb{R}^d$ *be open nonempty,* $\Lambda \in \mathcal{D}(\Omega)^*$*,* $\alpha \in \mathbb{N}_0^d$ *and* $f \in C^\infty(\Omega)$*. Then the following holds:*

(a) $D^\alpha\Lambda \in \mathcal{D}(\Omega)^*$.

(b) $f\Lambda \in \mathcal{D}(\Omega)^*$.

(c) For $g \in L_1^{\mathrm{loc}}(\Omega)$ *we have* $f\Lambda_g = \Lambda_{fg}$.

(d) For $g \in C^{|\alpha|}(\Omega)$ *we have* $D^\alpha\Lambda_g = \Lambda_{D^\alpha g}$.

Theorem 72. *Let* $\Omega \subset \mathbb{R}^d$ *be open nonempty, connected and* $\Lambda \in \mathcal{D}(\Omega)^*$ *be such that* $D^\alpha\Lambda = 0$ *for every multiindex* α *satisfying* $|\alpha| = 1$*. Then there exists* $c \in \mathbb{K}$ *such that* $\Lambda = \Lambda_c$.

Remark: proof was given only for the case $d = 1$ and $\Omega = (a, b)$. **The end of lecture 14**

Definition 73. Let $\Omega \subset \mathbb{R}^d$ be open nonempty. By the *space of distributions* we understand the locally convex space $(\mathcal{D}(\Omega)^*, w^*)$.

Proposition 74. *Let* $\Omega \subset \mathbb{R}^d$ *be open nonempty. Then the following holds:*

(a) If sequence $\{\Lambda_n\} \subset \mathcal{D}(\Omega)^*$ *converges to* $\Lambda \in \mathcal{D}(\Omega)^*$*, then*

- $D^\alpha\Lambda_n \to D^\alpha\Lambda$ *for every multiindex* $\alpha \in \mathbb{N}_0^d$*,*
- $f\Lambda_n \to f\Lambda$ *for every function* $f \in C^\infty(\Omega)$.

(b) Suppose we have functions $f_n, f \in L_1^{\mathrm{loc}}(\Omega)$ satisfying $\int_K |f_n - f| \,\mathrm{d}\lambda \to 0$ for every compact $K \subset \Omega$. Then $\Lambda_{f_n} \to \Lambda_f$.

(c) If $1 \leqslant p \leqslant \infty$ and $f_n \to f$ in $L_p(\Omega)$, then $\Lambda_{f_n} \to \Lambda_f$.

(d) If $\varphi_n \to \varphi$ in $\mathcal{D}(\Omega)$, then $\Lambda_{\varphi_n} \to \Lambda_\varphi$.

Proposition 75. *Let $\Omega \subset \mathbb{R}^d$ be open nonempty and (Λ_n) be sequence of distributions on Ω such that for every $\varphi \in \mathcal{D}(\Omega)$ the sequence $\big(\Lambda_n(\varphi)\big)$ is convergent. Then the functions $\Lambda : \mathcal{D}(\Omega) \to \mathbb{K}$ given by formula $\Lambda(\varphi) := \lim_{n\to\infty} \Lambda_n(\varphi)$, $\varphi \in \mathcal{D}(\Omega)$ is distribution on Ω.*

Definition 76. Let $f\colon \mathbb{R}^d \to \mathbb{K}$. Then we define the rotation of f as a function $\check{f}\colon \mathbb{R}^d \to \mathbb{K}$ given by the formula $\check{f}(x) = f(-x)$ pro $x \in \mathbb{R}^d$.

Definition 77. Let Λ be distribution on $\mathbb{R}^d$, $y \in \mathbb{R}^d$ and $\varphi \in \mathcal{D}(\mathbb{R}^d)$. Then we define shift of the distribution Λ as a distribution $\tau_y\Lambda$ given by the formula $\tau_y\Lambda(\psi) = \Lambda(\tau_{-y}\psi)$, $\psi \in \mathcal{D}(\mathbb{R}^d)$. Moreover, we define *convolution* of the function φ and distribution Λ by the formula $\Lambda * \varphi(x) := \Lambda(\tau_x\check{\varphi})$, $x \in \mathbb{R}^d$.

Theorem 78 (about convolution of a distribution with a function)**.** *Let Λ be a distribution on $\mathbb{R}^d$, $y \in \mathbb{R}^d$ and $\varphi, \psi \in \mathcal{D}(\mathbb{R}^d)$.*

(a) If $f \in L_1^{\mathrm{loc}}(\mathbb{R}^d)$, then $\Lambda_f * \varphi = f * \varphi$.

(b) $\Lambda * \varphi \in C^\infty(\mathbb{R}^d)$ and for every multiindex $\alpha \in \mathbb{N}_0^d$ we have $D^\alpha(\Lambda * \varphi) = D^\alpha\Lambda * \varphi = \Lambda * D^\alpha\varphi$.

(c) $\tau_y(\Lambda * \varphi) = \tau_y\Lambda * \varphi = \Lambda * \tau_y\varphi$.

(d) For $x_0 \in \mathbb{R}^d$ we have $\Lambda_{\delta_{x_0}} * \varphi = \tau_{x_0}\varphi$. In particular, $\Lambda_{\delta_0} * \varphi$.

The end of lecture 15

Definition 79. Let U be distribution on $\mathbb{R}^d$. Then we define rotation U as a distribution $\check{U}$ on $\mathbb{R}^d$ given by the formula $\check{U}(\varphi) = U(\check{\varphi})$, $\varphi \in \mathcal{D}(\mathbb{R}^d)$.

Examples 80. We try to define convolution of two distributions by the formula $U * V(\varphi) = U(\check{V} * \varphi)$, $\varphi \in \mathcal{D}(\mathbb{R}^d)$. But, since $\check{V} * \varphi$ sometimes does not have a compact support, we need to interpret this formula correctly. Several basic possible ways are mentioned bellow.

(i) For $f \in L_1^{\mathrm{loc}}(\mathbb{R}^d)$, $\varphi, \psi \in \mathcal{D}(\mathbb{R}^d)$ and a multiindex $\alpha \in \mathbb{N}_0^d$ we put

$$(D^\alpha\Lambda_f) * \Lambda_\varphi(\psi) := D^\alpha\Lambda_f(\widecheck{\Lambda_\varphi} * \psi).$$

Then $D^\alpha\Lambda_f * \Lambda_\varphi$ is a distribution and we have $(D^\alpha\Lambda_f) * \Lambda_\varphi = D^\alpha\Lambda_{f*\varphi} = D^\alpha(\Lambda_f * \Lambda_\varphi)$.

(ii) Given $x_0 \in \mathbb{R}^d$ and a distribution U on $\mathbb{R}^d$, for $\alpha \in \mathbb{N}_0^d$ and $\psi \in \mathcal{D}(\mathbb{R}^d)$ we define

$$U * D^\alpha\Lambda_{\delta_{x_0}}(\psi) := U(\widecheck{D^\alpha\Lambda_{\delta_{x_0}}} * \psi), \qquad D^\alpha\Lambda_{\delta_{x_0}} * U(\psi) := D^\alpha\delta_{x_0}(\check{U} * \psi),$$

where $D^\alpha\Lambda_{\delta_{x_0}}$ is understood as a functional from $(C^\infty(\mathbb{R}^d))^*$ defined by the formula $D^\alpha\Lambda_{\delta_{x_0}}(f) = (-1)^{|\alpha|}D^\alpha f(x_0)$ for $f \in C^\infty(\mathbb{R}^d)$. Then $U * D^\alpha\Lambda_{\delta_{x_0}} = D^\alpha\Lambda_{\delta_{x_0}} * U = \tau_{x_0}D^\alpha U$ and so the above are well-defined distributions. Moreover, we have $D^\alpha(U * \Lambda_{\delta_{x_0}}) = D^\alpha U * \Lambda_{\delta_{x_0}} = U * D^\alpha\Lambda_{\delta_{x_0}}$.

(iii) If $f, g, f * g \in L_1^{\mathrm{loc}}(\mathbb{R}^d)$, we put for every $\alpha \in \mathbb{N}_0^d$

$$D^\alpha\Lambda_f * \Lambda_g(\psi) := D^\alpha\Lambda_f(\widecheck{\Lambda_g} * \psi), \qquad \psi \in \mathcal{D}(\mathbb{R}^d),$$

where in the formula above we understand $D^\alpha\Lambda_f$ as a linear functional defined by the formula $D^\alpha\Lambda_f(h) = (-1)^{|\alpha|}\int_{\mathbb{R}^d} fD^\alpha h$ whenever $h \in C^\infty(\mathbb{R}^d)$ and $\int_{\mathbb{R}^d} fD^\alpha h$ is convergent. Then $D^\alpha\Lambda_f * \Lambda_g = D^\alpha\Lambda_{f*g}$ and so the above are well-defined distributions satisfying $D^\alpha(\Lambda_f * \Lambda_g) = (D^\alpha\Lambda_f) * \Lambda_g$.

(iv) If $f, g \in L_1^{\mathrm{loc}}(\mathbb{R}^d)$ are such that $(\operatorname{supp} f \cup \operatorname{supp} g) \subset (\mathbb{R}_+)^d$, then $f * g \in L_1^{\mathrm{loc}}(\mathbb{R}^d)$.

Definition 81. *Aproximative unit in $\mathcal{D}(\mathbb{R}^d)$* is a sequence of functions $(h_j)_{j=1}^\infty$ in $\mathcal{D}(\mathbb{R}^d)$, satisfying $h_j(x) = j^d h(jx)$ for $x \in \mathbb{R}^d$ and $j \in \mathbb{N}$, where $h \in \mathcal{D}(\mathbb{R}^d)$ is nonnegative function and $\int_{\mathbb{R}^d} h = 1$.

Proposition 82. *Let (h_j) be aproximative unit in $\mathcal{D}(\mathbb{R}^d)$, $\varphi \in \mathcal{D}(\mathbb{R}^d)$ and U be distribution on $\mathbb{R}^d$. Then $\varphi * h_j \to \varphi$ in the space $\mathcal{D}(\mathbb{R}^d)$ and $\Lambda_{U*h_j} \to U$ in the space $\mathcal{D}(\mathbb{R}^d)^*$.*

Remark: proof was given only for the "$\varphi * h_j \to \varphi$" part.**The end of lecture 16**

3. Tempered distributions

For $N \in \mathbb{N}_0$ and $f \in \mathcal{S}_d$ put
$$\nu_N(f) = \max_{|\alpha| \leqslant N} \left\| x \mapsto (1 + \|x\|^2)^N D^\alpha f(x) \right\|_\infty.$$
The metrizable locally convex topology on $\mathcal{S}_d$ generated by the system $\{\nu_N\}_{N=0}^\infty$ we denote as σ.

Theorem 83. *$(\mathcal{S}_d, \sigma)$ is Fréchet space and the topology σ has the following properties:*

(a) Let $\{f_n\}$ be a sequence in $\mathcal{S}_d$ and $f \in \mathcal{S}_d$. Then the following conditions are equivalent:

- *(i) $f_n \to f$ in topology σ.*
- *(ii) For every $N \in \mathbb{N}_0$ and every multiindex α of length d it holds that $(1 + \|x\|^2)^N D^\alpha f_n \to (1 + \|x\|^2)^N D^\alpha f$ uniformly on $\mathbb{R}^d$.*
- *(iii) For every polynomial P and every multiindex α of length d it holds that $PD^\alpha f_n \to PD^\alpha f$ uniformly on $\mathbb{R}^d$.*

(b) If $f_n \to f$ in the space $(\mathcal{S}_d, \sigma)$, then $f_n \to f$ in $L_p(\mathbb{R}^d)$ for every $1 \leqslant p < \infty$.

(c) If α is a multiindex of length d, P polynom on $\mathbb{R}^d$ and $g \in \mathcal{S}_d$, then mappings $f \mapsto D^\alpha f$, $f \mapsto Pf$ a $f \mapsto gf$ are continuous as maps from $(\mathcal{S}_d, \sigma)$ to $(\mathcal{S}_d, \sigma)$.

(d) For any compact $K \subset \mathbb{R}^d$ we have $\sigma\restriction_{\mathcal{D}(K)} = \tau_K$.

Proposition 84. *Subspace $\mathcal{D}(\mathbb{R}^d)$ is dense in $(\mathcal{S}_d, \sigma)$ and for the topology τ we have $\sigma\restriction_{\mathcal{D}(\mathbb{R}^d)} \subset \tau$. In other words, embedding $Id\colon (\mathcal{D}(\mathbb{R}^d), \tau) \to (\mathcal{S}_d, \sigma)$ is continuous onto a dense set.*

Definition 85. Distributions on $\mathbb{R}^d$, which are restrictions of functionals from $(\mathcal{S}_d, \sigma)^*$ are called *tempered distributions*.

Proposition 86. *Distribution Λ on $\mathbb{R}^d$ is tempered if and only if there are $N \in \mathbb{N}_0$ and $C > 0$ satisfying $|\Lambda(\varphi)| \leqslant C\nu_N(\varphi)$, $\varphi \in \mathcal{D}(\mathbb{R}^d)$.*

Examples 87. (a) Every distribution with a compact support (that is, satisfying that there exists a compact $K \subset \mathbb{R}^d$ such that $\Lambda(\varphi) = 0$ whenever $\varphi \in \mathcal{D}(\mathbb{R}^d \setminus K)$) is tempered.

(b) Whenever μ is borel measure satisfying $\int (1 + \|x\|^2)^{-N} \, \mathrm{d}\mu < \infty$ for some $N \in \mathbb{N}_0$, then Λ_μ is tempered distribution and $\Lambda_\mu(f) = \int_{\mathbb{R}^d} f \, \mathrm{d}\mu$ for $f \in \mathcal{S}_d$.

(c) Whenever g is measurable function on $\mathbb{R}^d$ such that $x \mapsto (1 + \|x\|^2)^N \in L_p(\mathbb{R}^d)$ for some $N \in \mathbb{N}_0$ and $1 \leqslant p \leqslant \infty$ (This in particular holds for functions from $L_p(\mathbb{R}^d)$ or for functions majorizable by a polynom). Then Λ_g is tempered distribution, where $\Lambda_g(f) = \int_{\mathbb{R}^d} fg$ for $f \in \mathcal{S}_d$.

Proposition 88. *Let Λ be a tempered distribution on $\mathbb{R}^d$, $\alpha \in \mathbb{N}_0^d$, $g \in \mathcal{S}_d$ and P be polynomial on $\mathbb{R}^d$. Then $D^\alpha \Lambda$, $g\Lambda$ and $P\Lambda$ are tempered distributions as well and the following formulas hold for every $f \in \mathcal{S}_d$:*

- *$D^\alpha \Lambda(f) = (-1)^{|\alpha|} \Lambda(D^\alpha f)$,*
- *$(g\Lambda)(f) = \Lambda(gf)$ and*
- *$(P\Lambda)(f) = \Lambda(Pf)$.*

Moreover, mappings $\Lambda \mapsto D^\alpha \Lambda$, $\Lambda \mapsto g\Lambda$ and $\Lambda \mapsto P\Lambda$ are continuous mappings from the space $(\mathcal{S}_d^*, w^*)$ into itself.

The end of lecture 17

Theorem 89. *Fourier transform is isomorfism $\mathcal{S}_d$ onto $\mathcal{S}_d$. Moreover, for every $f \in \mathcal{S}_d$ we have*
$$\hat{\hat{f}}(x) = f(-x) \text{ pro každé } x \in \mathbb{R}^d \quad \text{and} \quad \hat{\hat{\hat{\hat{f}}}} = f.$$

Corollary 90. *For $f, g \in \mathcal{S}_d$ we have $(2\pi)^{d/2} \widehat{fg} = \hat{f} * \hat{g}$. In particular, the space $\mathcal{S}_d$ is closed under the operation of convolution.*

Definition 91. *Fourier transformation of tempered distribution Λ on $\mathbb{R}^d$* is defined by the formula $\widehat{\Lambda}(f) = \Lambda(\hat{f})$ for $f \in \mathcal{S}_d$.

Theorem 92.

(a) If $g \in L_1(\mathbb{R}^d)$, then $\Lambda_{\hat{g}}$ is tempered distribution and $\widehat{\Lambda_g} = \Lambda_{\hat{g}}$. If $g \in L_2(\mathbb{R}^d)$, then $\widehat{\Lambda_g} = \Lambda_{F(g)}$, where F is the extension of the Fourier transformation from the Plancherel theorem.

(b) If Λ is tempered distribution on $\mathbb{R}^d$ a $\alpha \in \mathbb{N}_0^d$, then

- $\widehat{D^\alpha \Lambda} = s_\alpha \widehat{\Lambda}$, *where* $s_\alpha(x) = (ix)^\alpha$, *a*
- $D^\alpha \widehat{\Lambda} = \widehat{m_\alpha \Lambda}$, *where* $m_\alpha(x) = (-ix)^\alpha$.

(c) Fourier transformation $\mathcal{F}$ *of tempered distributions is isomorphism of the space* $(\mathcal{S}_d^*, w^*)$ *onto itself. Moreover, we have* $\mathcal{F}^4 = Id$.

Definition 93. For $\Lambda \in \mathcal{S}(\mathbb{R}^d)^*$ and function $f \in \mathcal{S}(\mathbb{R}^d)$ we define the mapping $\Lambda * \Lambda_\varphi : \mathcal{S}(\mathbb{R}^d) \to \mathbb{K}$ by the formula

$$\Lambda * \Lambda_f(g) := \Lambda(\check{f} * g), \qquad g \in \mathcal{S}(\mathbb{R}^d).$$

Proposition 94. *Let* $\Lambda \in \mathcal{S}(\mathbb{R}^d)^*$ *and* $f \in \mathcal{S}(\mathbb{R}^d)$. *Then the following holds.*

(a) $\Lambda * \Lambda_f \in \mathcal{S}(\mathbb{R}^d)^*$ *and for every* $\alpha \in \mathbb{N}_0^d$ *we have* $D^\alpha(\Lambda * \Lambda_f) = (D^\alpha \Lambda) * \Lambda_f$.

(b) $\Lambda_{\delta_0} * \Lambda_f = \Lambda_f$.

The end of lecture 18

III. Basics on vector integration

1. Measurable mappings

Definition 95. Let $(\Omega, \mathcal{A})$ be a measurable space and X a Banach space. Mapping $f\colon \Omega \to X$ is

- *borel $\mathcal{A}$-measurable*, if $f^{-1}(U) \in \mathcal{A}$ for every $U \subset X$ open,
- *simple measurable*, if $f(\Omega)$ is a finite set and f is borel $\mathcal{A}$-measurable (that is, $f = \sum_{i=1}^n x_i \chi_{A_i}$ where $(x_i)_{i=1}^n \in X^n$ and $A_1, \dots, A_n \in \mathcal{A}$ are pairwise disjoint),
- *strongly $\mathcal{A}$-measurable*, if f is a pointwise limit of simple measurable functions (that is, there are simple measurable functions (s_n) such that $\|s_n(x) - f(x)\| \to 0$ for every $x \in \Omega$),
- *weakly $\mathcal{A}$-measurable*, if for every $x^* \in X^*$ the mapping $x^* \circ f : \Omega \to \mathbb{K}$ is borel $\mathcal{A}$-measurable.

Proposition 96. *Let* $(\Omega, \mathcal{A})$ *be a measurable space and* X *a Banach space.*

(a) Pointwise limit of a sequence of borel $\mathcal{A}$*-measurable mappings from* Ω *to* X *is borel* $\mathcal{A}$*-measurable mapping.*

(b) For function $f : \Omega \to X$ *we have*

$$f \text{ is strongly } \mathcal{A}\text{-measurable} \Rightarrow f \text{ is borel } \mathcal{A}\text{-measurable} \Rightarrow f \text{ is weakly } \mathcal{A}\text{-measurable}$$

(c) If X *is separable, then* $f : \Omega \to X$ *is strongly* $\mathcal{A}$*-measurable, if and only if it is borel* $\mathcal{A}$*-measurable.*

(d) Pointwise limit of a sequence of strongly (resp. weakly) $\mathcal{A}$*-measurable mappings from* Ω *to* X *is strongly (resp. weakly)* $\mathcal{A}$*-measurable mapping.*

(e) Simple measurable, strongly $\mathcal{A}$*-measurable and weakly* $\mathcal{A}$*-measurable mappings form a vector space.*

Definition 97. Let $(\Omega, \mathcal{A}, \mu)$ be a space with a complete measure and X be a Banach space. Mapping $f\colon \Omega \to X$ is *strongly μ-measurable*, if f is μ-a.e. pointwise limit of a sequence of simple measurable functions (that is, there are simple measurable functions (s_n) such that $\|s_n(x) - f(x)\| \to 0$ for μ-a.e. $x \in \Omega$). Říkáme, že $f\colon \Omega \to X$ je *slabě μ-measurable* (resp. *borelovsky μ-measurable*), pokud je slabě $\mathcal{A}$-měřitelné (resp. borelovsky $\mathcal{A}$-measurable).

Lemma 98. *Let* $(\Omega, \mathcal{A}, \mu)$ *be a space with a complete measure,* X *a Banach space and* $f\colon \Omega \to X$. *Then* f *is strongly* μ*-measurable, if and only if* f *is borel* μ*-measurable and there is* $E \subset \Omega$ *such that* $\mu(E) = 0$ *and* $f(\Omega \backslash E)$ *is separable.*

Corollary 99. *Let* $(\Omega, \mathcal{A}, \mu)$ *be a space with a complete measure,* X *a Banach space and* $f_n\colon \Omega \to X$, $n \in \mathbb{N}$ *is sequence of strongly* μ*-measurable mappings, which pointwise converges a.e. to* $f\colon \Omega \to X$. *Then* f *is strongly measurable.*

The end of lecture 19

Theorem 100 (Pettis theorem)**.** *Let* $(\Omega, \mathcal{A}, \mu)$ *be a space with a complete measure,* X *a Banach space and* $f\colon \Omega \to X$. *Then the following assertions are equivalent:*

(i) f *is strongly* μ*-measurable.*

(ii) f *is borel* μ*-measurable and there is* $E \subset \Omega$ *such that* $\mu(E) = 0$ *and* $f(\Omega \backslash E)$ *is separable.*

(iii) f *is weakly* μ*-measurable and there is* $E \subset \Omega$ *such that* $\mu(E) = 0$ *and* $f(\Omega \backslash E)$ *is separable.*

2. Dunford and Pettis integral

Definition 101. Let $(\Omega, \mathcal{A}, \mu)$ be a space with a complete measure and X be a Banach space. Function $f : \Omega \to X$ is *weakly integrable*, if for every $x^* \in X^*$ we have $x^* \circ f \in L_1(\mu)$.

If $f : \Omega \to X$ is weakly integrable and $E \subset \Omega$ is measurable, then *Dunford integral f over the set E* is the point $(D)\int_E f \, \mathrm{d}\mu \in X^{**}$ satisfying

$$\Big((D)\int_E f \, \mathrm{d}\mu\Big)(x^*) = \int_E x^* \circ f \, \mathrm{d}\mu, \quad x^* \in X^*.$$

Proposition 102. *Let $(\Omega, \mathcal{A}, \mu)$ be a space with a complete measure, X be a Banach space, $f : \Omega \to X$ be weakly integrable and $E \subset \Omega$ measurable. Then there is a unique $x^{**} \in X^{**}$ such that x^{**} is Dunford integral of f over the set E.*

Definition 103. Let $(\Omega, \mathcal{A}, \mu)$ be a space with a complete measure, X be a Banach space and $f : \Omega \to X$ be weakly integrable. If $(D)\int_E f \, \mathrm{d}\mu \in X$ (or more precisely $(D)\int_E f \, \mathrm{d}\mu \in \varepsilon(X) \subset X^{**}$) for every $E \subset \Omega$ measurable, then we say that f is *Pettis integrable* and

$$(P)\int_E f \, \mathrm{d}\mu = (D)\int_E f \, \mathrm{d}\mu$$

is the *Pettis integral of f over the set E*.

The end of lecture 20

3. Bochner integral

Definition 104. Let (Ω, μ) be a space with a measure and X be a Banach space. Simple, measurable function $f\colon \Omega \to X$ is *Bochner integrable*, if for every $x \in f(\Omega)\backslash\{0\}$ we have $\mu(f^{-1}(x)) < +\infty$.

If $f\colon \Omega \to X$ is simple, measurable and Bochner integrable, then for every measurable $E \subset \Omega$ we define the *Bochner integral* of f over the set E as

$$(B)\int_E f \, \mathrm{d}\mu = \sum_{x \in f(\Omega)\backslash\{0\}} \mu(f^{-1}(x) \cap E)x.$$

Lemma 105. *Let $(\Omega, \mathcal{A}, \mu)$ be a space with a complete measure and X be a Banach space.*

(i) Bochner integrable simple functions form a vector space and the mapping which assigns to a simple integrable function f its integral $(B)\int_\Omega f \, \mathrm{d}\mu$, is linear.

(ii) If $f\colon \Omega \to X$ is simple, measurable, then f is Bochner integrable, if and only if the function $t \mapsto \|f(t)\|$ is integrable. In this case $\|(B)\int_E f \, \mathrm{d}\mu\| \leqslant \int_E \|f\| \, \mathrm{d}\mu$ for every measurable $E \subset \Omega$.

Definition 106. Let $(\Omega, \mathcal{A}, \mu)$ be a space with a complete measure, X be a Banach space and $f : \Omega \to X$ be strongly μ-measurable. We say that f is *bochner integrable*, if there exists a sequence $f_n\colon \Omega \to X$, $n \in \mathbb{N}$ of simple Bochner integrable mappings such that $\lim_{n\to\infty} \int_\Omega \|f_n - f\| \, \mathrm{d}\mu = 0$. Then, for every measurable $E \subset \Omega$ we define the *Bochner integral* of f over E as

$$(B)\int_E f \, \mathrm{d}\mu = \lim_{n\to\infty} (B)\int_E f_n \, \mathrm{d}\mu.$$

Theorem 107. *Let $(\Omega, \mathcal{A}, \mu)$ be a space with a complete measure and X be a Banach space.*

(a) The limit defining the Bochner integral exists and does not depend of the choice of the sequence (f_n).

(b) Bochner integrable functons form a vector space and the mapping which assigns to every bochner integrable function f its integral $(B)\int_\Omega f \, \mathrm{d}\mu$, is linear.

(c) $\|(B)\int_E f \, \mathrm{d}\mu\| \leqslant \int_E \|f\| \, \mathrm{d}\mu$ for every $E \subset \Omega$ measurable and $f : \Omega \to X$ bochner integrable.

Theorem 108. *Let $(\Omega, \mathcal{A}, \mu)$ be a space with a complete measure, X be a Banach space and $f : \Omega \to X$ be strongly μ-measurable. Then f is bochner integrable, if and only if $\|f\|$ is lebesgue integrable.*

Theorem 109 (about majorizable convergence)**.** *Let $(\Omega, \mathcal{A}, \mu)$ be a space with a complete measure, X be a Banach space and $f_n : \Omega \to X$, $n \in \mathbb{N}$ be sequence of strongly μ-measurable mappings. Let $f : \Omega \to X$ be such that $f_n \to f$ pointwise a.e., ane let $g \in L_1(\mu)$ be such that for every $n \in \mathbb{N}$ we have $\|f_n(t)\| \leqslant g(t)$ for a. e. $t \in \Omega$. Then f is bochner integrable and $(B)\int_\Omega f \, \mathrm{d}\mu = \lim\limits_{n\to\infty} (B)\int_\Omega f_n \, \mathrm{d}\mu$.*

Theorem 110 (absolute continuuity of Bochner integral)**.** *Let $(\Omega, \mathcal{A}, \mu)$ be a space with a complete measure, X be a Banach space and $f : \Omega \to X$ be bochner integrable. Then for every $\varepsilon > 0$ there exists $\delta > 0$ such that $\|(B)\int_E f \, \mathrm{d}\mu\| < \varepsilon$ whenever $E \subset \Omega$ is such that $\mu(E) < \delta$.*

Theorem 111. *Let $(\Omega, \mathcal{A}, \mu)$ be a space with a complete measure, X and Y be Banach spaces, $f : \Omega \to X$ bochner integrable and $T \in \mathcal{L}(X, Y)$. Then $T \circ f$ is bochner integrable and for every measurable $E \subset \Omega$ we have*

$$(B)\int_E T \circ f \,\mathrm{d}\mu = T\Big((B)\int_E f \,\mathrm{d}\mu\Big).$$

In particular, f is Pettis integrable and $(P)\int_E f \,\mathrm{d}\mu = (B)\int_E f \,\mathrm{d}\mu$ for every measurable $E \subset \Omega$.

The end of lecture 21

4. Lebesgue-Bochner spaces

Definition 112. Let $(\Omega, \mathcal{A}, \mu)$ be a space with a complete measure, X a Banach space and $1 \leqslant p \leqslant \infty$. By $L_p(\mu, X)$ we denote the set of all strongly measurable mappings from Ω into X such that $\|f\| \in L_p(\mu)$, factorized by the equality μ-a. e.

Moreover, for $f \in L_p(\mu, X)$ we denote $\|f\|_{L_p(\mu,X)} = \big\|t \mapsto \|f(t)\|\big\|_{L_p(\mu)}$.

Theorem 113. *Let $(\Omega, \mathcal{A}, \mu)$ be a space with a complete measure, X a Banach space and $1 \leqslant p \leqslant \infty$.*

(a) $L_p(\mu, X)$ is Banach space with norm $\|f\|_{L_p(\mu,X)}$.

(b) If X is a Hilbert space, then $L_2(\mu, X)$ is Hilbert space with the scalar product

$$\langle f, g\rangle_{L_2(\mu,X)} = \int_\Omega \langle f(t), g(t)\rangle \,\mathrm{d}\mu.$$

Remark: the proof of part (b) was omitted

Theorem 114. *Let $(\Omega, \mathcal{A}, \mu)$ be a space with a complete measure, X a Banach space and $1 \leqslant p \leqslant \infty$.*

(a) The set of simple Bochner integrable mappings from Ω into X is dense in $L_p(\mu, X)$.

(b) If X and $L_p(\mu)$ are separable, then $L_p(\mu, X)$ is separable.

The end of lecture 22

Theorem 115. *Let $(\Omega, \mathcal{A}, \mu)$ be a space with a complete measure, X a Banach space, $1 \leqslant p \leqslant \infty$ and q be conjugate exponent to p. Consider the mapping $I\colon L_q(\mu, X^*) \to L_p(\mu, X)^*$, $I(g) = \varphi_g$, where*

$$\varphi_g(f) = \int_\Omega g(t)\big(f(t)\big) \,\mathrm{d}\mu(t), \quad f \in L_p(\mu, X).$$

Then the following holds.

(a) The mapping I is isometry.

(b) If (Ω, μ) is atomic and $p \neq 1$, then I is onto. In particular, $\ell_q(J, X^*)$ is isometric to $\ell_p(J, X)^*$ for every set J. Moreover, if μ is σ-additive measure, then the same holds even for $p = 1$.

(c) If X is reflexive and $p \neq 1$, then I is onto. If μ is moreover σ-additive measure, then the same holds even for $p = 1$.

Remark: we proved only the part (a) for $p \neq 1$

Theorem 116. *Let $(\Omega, \mathcal{A}, \mu)$ be a space with a complete measure, X be reflexive Banach space and $1 < p < \infty$. Then $L_p(\mu, X)$ is reflexive.*

IV. Convex compact sets

Definition 117. Let C be convex subset of a vector space. We say that nonempty $F \subset C$ is *extremal subset of C*, if no point of F is a nontrivial convex combination of points from C, some of which is not in E, that is, if $\lambda x + (1-\lambda)y \in F$ for some $x, y \in C$ and $\lambda \in (0, 1)$, then $x, y \in F$.

We say $x \in C$ is an *extreme point of the set C*, if $\{x\}$ is extremal subset of C. The set of all the extreme points of C is denoted as $\operatorname{ext} C$.

Fact 118. *Let C be a convex set in a vector space and $F \subset C$. Then the following conditions are equivalent.*

(a) F is an extreme subset of C.

(b) If $\frac{1}{2}(x + y) \in F$ for some $x, y \in C$, then $x, y \in F$.

The end of lecture 23

Theorem 119 (Krein-Milman)**.** *Let X be HLCS and let $K \subset X$ be compact and convex. Then $K = \overline{\operatorname{conv}} \operatorname{ext} K$.*

Definition 120. Let $A \subset \mathbb{R}^d$. By $\operatorname{cone}(A)$ we denote the set of all nonnegative linear combinations of points from A, where nonnegative (resp. positive) linear combination of points $x_1, \dots, x_m \in A$ is of the form $\sum_{i=1}^m \alpha_i x_i$, where $\alpha \geqslant 0$ (resp. $\alpha > 0$) for every $i \leqslant m$.

Theorem 121. *Let $A \subset \mathbb{R}^d$.*

(a) Every nonzero vector from $\operatorname{cone}(A)$ can be expressed as a positive linear combination of linearly independent vectors from A.

(b) Every vector from $\operatorname{conv}(A)$ can be expressed as a convex combination of $d+1$ vectors from A.

(c) If A is compact and convex subset, then every point $x \in A$ is a convex combination of at most $d+1$ extreme points of the set A.

The end of lecture 24

Example 122. Let K be a Hausdorff compact space and let $P(K)$ be Radon probability measures on K, that is $P(K) = \{\mu \in \mathcal{M}(K);\ \mu \geqslant 0, \mu(K) = 1\}$. Then $(P(K), w^*) \subset (\mathcal{M}(K), w^*)$ is a compact convex set and $\operatorname{ext} P(K) = \{\delta_x;\ x \in K\}$.

Definition 123. Let X be HLCS, $K \subset X$ be a compact convex set and $\mu \in P(K)$. A point $x \in K$ is the *barrycenter* of the measure μ (we write $x = r(\mu)$), if for everz continuous affine $f : K \to \mathbb{R}$ we have

$$f(x) = \int_K f \, \mathrm{d}\mu.$$

Proposition 124. *Let X be HLCS, $K \subset X$ be a compact convex set and $\mu \in P(K)$. Then there exists a unique barrycenter $r(\mu) \in K$ of the measure μ.*

Theorem 125 (integral representation)**.** *Let X be HLCS, $K \subset X$ be a compact convex set and $x \in K$. Then there exists $\mu \in P(K)$ satisfying $r(\mu) = x$ and $\mu(\overline{\operatorname{ext} K}) = 1$.*

The end of lecture 25