Mesh Refinement for Quasilinear Two-Grid Discontinuous Galerkin Finite Element Methods with Polygonal Meshes

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Overview

Standard Formulation

Nonlinear Problem

Given a semilinear form $\mathcal{N}(\cdot;\cdot,\cdot)$, find $u \in V$ such that

$$\mathcal{N}(u;u,v) = 0 \qquad \forall v \in V.$$

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(Standard) Discretization Method

Find $u_h \in V_h$ such that

$$\mathcal{N}_h(u_h;u_h,v_h) = 0 \qquad \forall v_h \in V_h.$$

Create a mesh which is 'coarser' than the original mesh and define V_H as the FE space on this mesh, then:

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Find $u_H \in V_H$ such that

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find $u_{2G} \in V_h$ such that

$$\mathcal{N}_h(u_H; u_{2G}, v_h) = 0 \qquad \forall v_h \in V_h.$$

Xu 1992, 1994, 1996, Xu & Zhou 1999, Axelsson & Layton 1996, Dawson, Wheeler & Woodward 1998, Utnes 1997, Marion & Xu 1995, Wu & Allen 1999, Bi & Ginting 2007, 2011

Second-Order Quasilinear PDEs

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Quasilinear Problem

Given $\Omega \subset \mathbb{R}^d$, $d = 2, 3$ and $f \in L^2(\Omega)$, find u such that

$$-\nabla \cdot \{\mu(\boldsymbol{x}, |\nabla u|)\nabla u\} = f \qquad \text{in } \Omega,$$
$$u = 0 \qquad \text{on } \Gamma.$$

Assumption

1. $\mu \in C(\bar{\Omega} \times [0, \infty))$ *and*
2. *there exists positive constants* m_μ *and* M_μ *such that*

$$M_\mu(t - s) \leq \mu(\boldsymbol{x}, t)t - \mu(\boldsymbol{x}, s)s \leq M_\mu(t - s), \quad t \geq s \geq 0, \quad \boldsymbol{x} \in \bar{\Omega}.$$

- $\mathcal{T}_h$ is a mesh consisting of triangles/tetrahedrons elements κ of granularity h, which are an affine map of a reference element $\hat{\kappa}$; i.e., there exists an affine mapping $T_\kappa : \hat{\kappa} \to \kappa$ such that $\kappa = T_\kappa(\hat{\kappa})$.
- Define polynomial degree k_κ for all $\kappa \in \mathcal{T}_h$
- (Fine) *hp*-DG finite element space:

$$V_{hk}(\mathcal{T}_h, \boldsymbol{k}) = \{v \in L^2(\Omega) : v|_\kappa \circ T_\kappa \in \mathcal{P}_{k_\kappa}(\hat{\kappa}), \kappa \in \mathcal{T}_h\}.$$

- $\mathcal{F}_h = \mathcal{F}_h^B \cup \mathcal{F}_h^I$ denotes the set of all faces in the mesh $\mathcal{T}_h$.
- Trace operators

$$\{\!\{\cdot\}\!\} : \text{Average Operator} \qquad [\![\cdot]\!] : \text{Jump Operator}.$$

(Standard) Interior Penalty Method

Find $u_{hk} \in V_{hk}(\mathcal{T}_h, \boldsymbol{k})$ such that

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$$\begin{aligned} A_{hk}(\psi; u, v) = &\int_\Omega \mu(|\nabla_h \psi|) \nabla_h u \cdot \nabla_h v \, d\boldsymbol{x} + \int_{\mathcal{F}_h} \sigma_{h,k} [\![u]\!] \cdot [\![v]\!] \, ds \\ &- \int_{\mathcal{F}_h} \{\!\{\mu(|\nabla_h \psi|) \nabla_h u\}\!\} \cdot [\![v]\!] \, ds, \\ &+ \theta \int_{\mathcal{F}_h} \{\!\{\mu(h_F^{-1} |[\![\psi]\!]|) \nabla_h v\}\!\} \cdot [\![u]\!] \, ds, \\ F_{hk}(v) = &\int_\Omega f v \, d\boldsymbol{x}. \end{aligned}$$

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Interior penalty parameter:

$$\sigma_{h,k} = \gamma \frac{k_F^2}{h_F},$$

where $k_F = \max(k_{\kappa_1}, k_{\kappa_2})$ and h_F is the diameter of the face.

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References:

Bustinza & Gatica 2004, Gatica, Gonzáles & Meddahi 2004, Houston, Robson & Suli 2005, Bustinza, Cockburn & Gatica 2005, Houston, Süli & Wihler 2007, Gudi, Nataraj & Pani 2008

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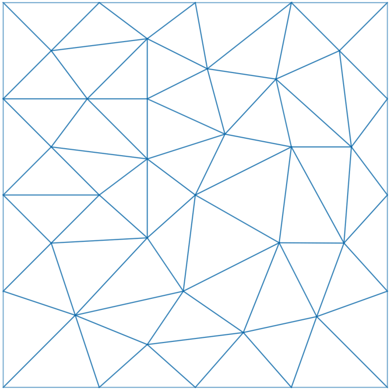

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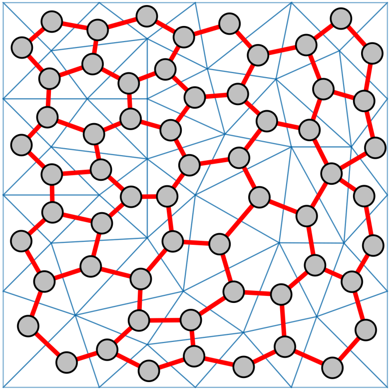

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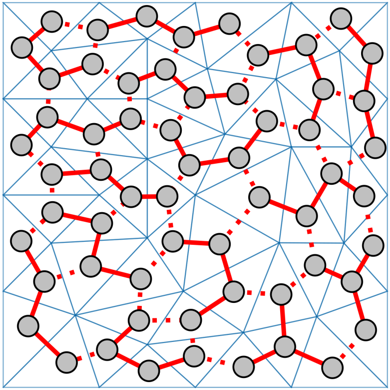

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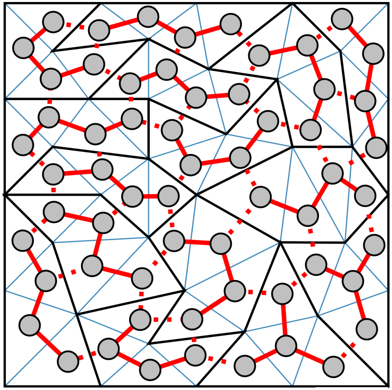

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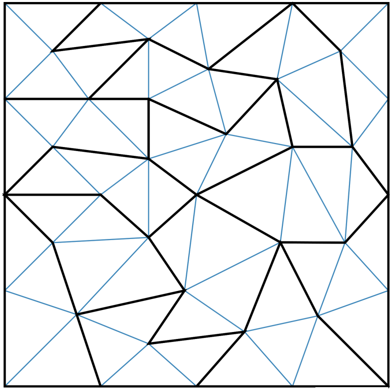

For example, METIS - Karypis & Kumar 1999

- Define $\mathcal{T}_h(\kappa_H) = \{\kappa \in \mathcal{T}_h : \kappa \subseteq \kappa_H\}$ for all $\kappa_H \in \mathcal{T}_H$.
- Define polynomial degree K_{κ_H}, for all $\kappa_h \in \mathcal{T}_H$, such that

$$K_{\kappa_H} \leq k_\kappa \text{ for all } \kappa \in \mathcal{T}_h(\kappa_H).$$

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Two-Grid *hp*-DGFEM

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- $V_{HK}(\mathcal{T}_H, \boldsymbol{K}) \subseteq V_{hk}(\mathcal{T}_h, \boldsymbol{k})$
- We use a different *interior penalty parameter*:

$$\sigma_{h,k} = \gamma \max_{\kappa \in \{\kappa_1, \kappa_2\}} C_{\text{inf}}(k_\kappa, \kappa, F) k_\kappa^2 |F| |\kappa|^{-1},$$

where

$$C_{\text{inf}}(k, \kappa, F) = \min \left\{ \frac{|\kappa|}{\sup_{\kappa_\flat^F \subset \kappa} |\kappa_\flat^F|}, k^{2d} \right\}.$$

$\kappa_\flat^F$ represents a simplex sharing an edge with κ)

[Cangiani, Georgoulis, & Houston]

Two-Grid Approximation

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3. Determine the fine grid approximation $u_{2G} \in V_{hk}(\mathcal{T}_h, \boldsymbol{k})$ such that

$$A_{hk}(u_{HK}; u_{2G}, v_{hk}) = F_{hk}(v_{hk})$$

   for all $v_{hk} \in V_{hk}(\mathcal{T}_h, \boldsymbol{k})$.

[C., Houston, & Wihler 2013]

Lemma (Standard Quasilinear DGFEM)

The following bound holds:

$$\|u - u_{hk}\|_{hk}^2 \leq C_1 \sum_{\kappa \in \mathcal{T}_h} \eta_\kappa^2 \qquad .$$

Here the local error indicators η_κ are defined, for all $\kappa \in \mathcal{T}_h$, as

$$\begin{aligned}\eta_\kappa^2 &= h_\kappa^2 k_\kappa^{-2} \|f + \nabla \cdot \{\mu(|\nabla u_{hk}|)\nabla u_{hk}\}\|_{L^2(\kappa)}^2 \\ &\quad + h_\kappa k_\kappa^{-1} \|[\![\mu(|\nabla u_{hk}|)\nabla u_{hk}]\!]\|_{L^2(\partial\kappa\setminus\Gamma)}^2 + \gamma^2 k_\kappa^3 h_\kappa^{-1} \|[\![u_{hk}]\!]\|_{L^2(\partial\kappa)}^2\end{aligned}$$

Proof.

See Houston, Süli & Wihler 2008. □

Lemma (Two-Grid Quasilinear Approximation)

The following bound holds:

$$\|u - u_{2G}\|_{hk}^2 \leq C_2 \sum_{\kappa \in \mathcal{T}_h} \left(\eta_\kappa^2 + \xi_\kappa^2 \right).$$

Here the local error indicators η_κ are defined, for all $\kappa \in \mathcal{T}_h$, as

$$\eta_\kappa^2 = h_\kappa^2 k_\kappa^{-2} \|f + \nabla \cdot \{\mu(|\nabla u_{HK}|)\nabla u_{2G}\}\|_{L^2(\kappa)}^2 + h_\kappa k_\kappa^{-1} \|[\![\mu(|\nabla u_{HK}|)\nabla u_{2G}]\!]\|_{L^2(\partial\kappa\setminus\Gamma)}^2 + \gamma^2 k_\kappa^3 h_\kappa^{-1} \|[\![u_{2G}]\!]\|_{L^2(\partial\kappa)}^2$$

and the local two-grid error indicators are defined, for all $\kappa \in \mathcal{T}_h$, as

$$\xi_\kappa^2 = \|(\mu(|\nabla u_{HK}|) - \mu(|\nabla u_{2G}|))\nabla u_{2G}\|_{L^2(\kappa)}^2.$$

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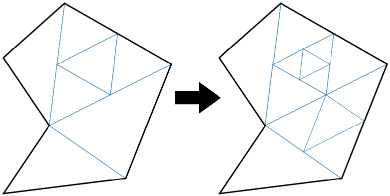
7. Goto 2.

The constants λ_F and λ_C are steering parameters.

Fine Element Refine:



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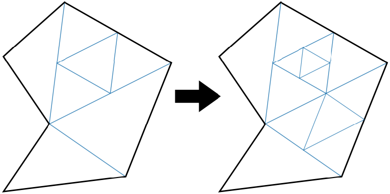
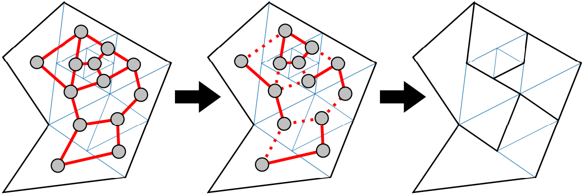


Coarse Element Refine — Partition patch of fine elements into 2^d elements



Similar to [Collis & Houston, 2016]

Coarse Element h-Refinement

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However, we have information about the error for each fine element — can we distribute the agglomeration using this information?

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Two options for weights:

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- Total Error: $\eta_\kappa + \xi_\kappa$

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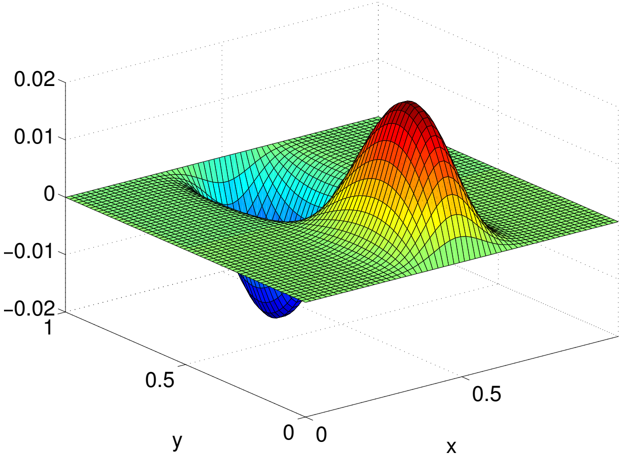
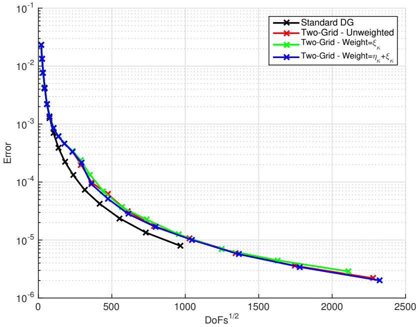
The coarse element refinement uses the fine elements *after* refinement; therefore, we divide the weight from a fine element marked for refinement equally between the new fine elements.

Quasilinear PDE: Smooth Solution

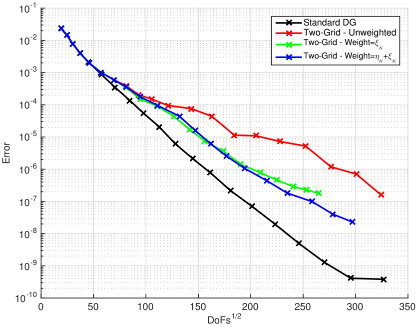
We let $\Omega = (0,1)^2, \mu(\boldsymbol{x}, |\nabla u|) = 2 + \frac{1}{1+|\nabla u|^2}$ and select f so that

$$u(x,y) = x(1-x)y(1-y)(1-2y)\mathrm{e}^{-20(2x-1)^2}.$$





Error vs. #DoFs
(*h*-refinement)



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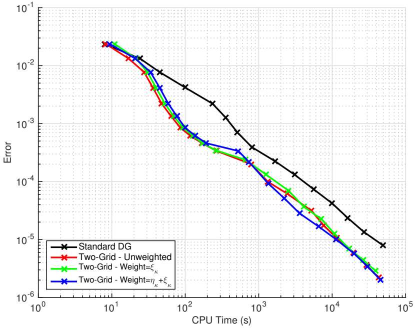
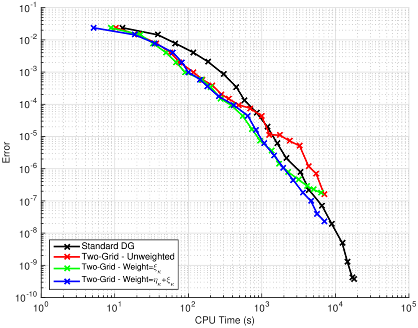
Quasilinear PDE: Smooth Solution



Error vs. Computation Time (*h*-refinement)



Error vs. Computation Time (*hp*-refinement)

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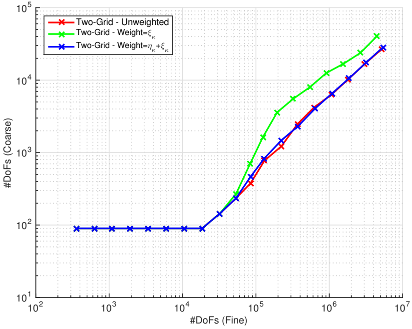
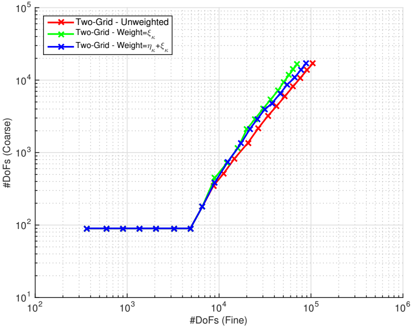


#DoFs: Fine vs. Coarse
(*h*-refinement)



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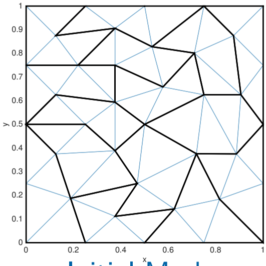
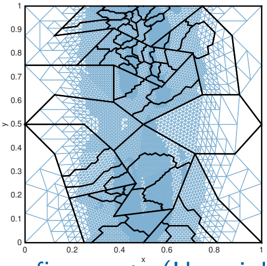
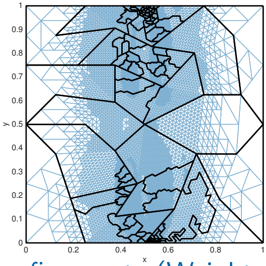


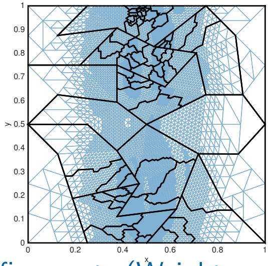
Initial Mesh



10 h-refinements (Unweighted)

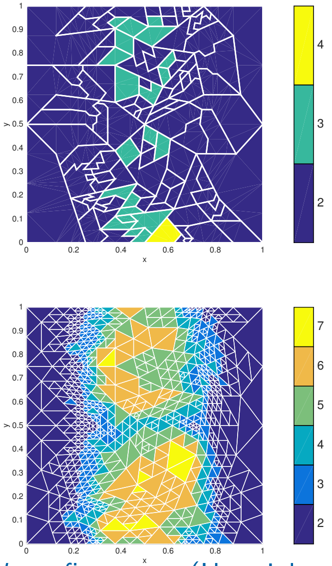


10 h-refinements (Weight $= \xi_\kappa$)

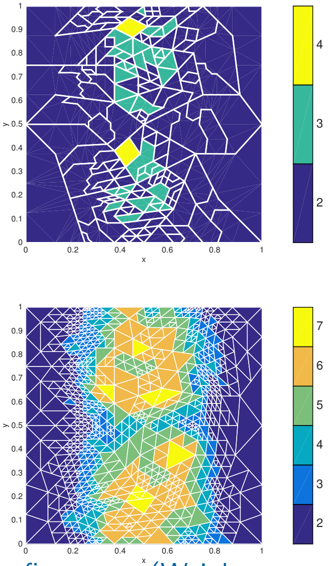


10 h-refinements (Weight $= \eta_\kappa + \xi_\kappa$)

Quasilinear PDE: Smooth Solution



10 *hp*-refinements (Unweighted)

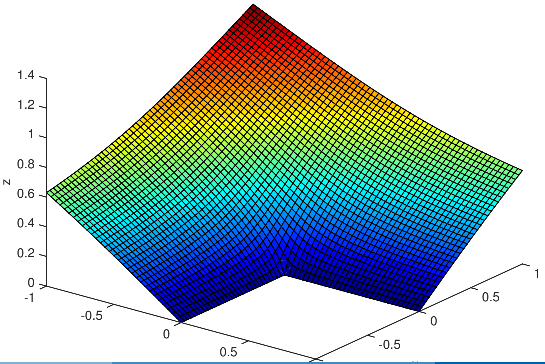


10 *hp*-refinements (Weight $= \eta_\kappa + \xi_\kappa$)

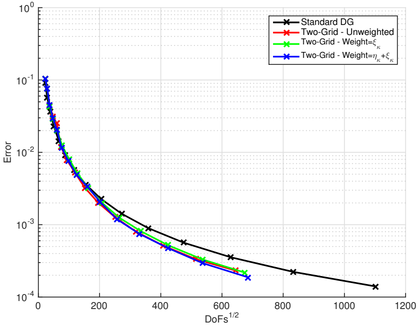
We let $\Omega = (-1,1)^2 \setminus [0,1) \times (-1,0]$, $\mu(\boldsymbol{x}, |\nabla u|) = 1 + \mathrm{e}^{-|\nabla u|^2}$ and select f so that

$$u(r,\phi) = r^{2/3} \sin\left(\frac{2}{3}\varphi\right).$$

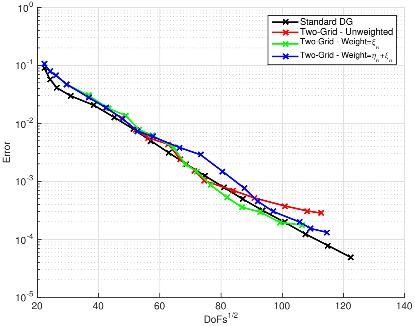
Note that u in analytic in $\overline{\Omega} \setminus \{\mathbf{0}\}$, but ∇u is singular at the origin.



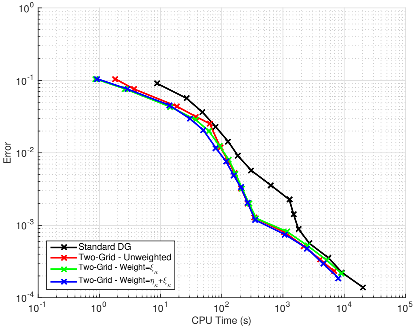
Quasilinear PDE: Singular Solution



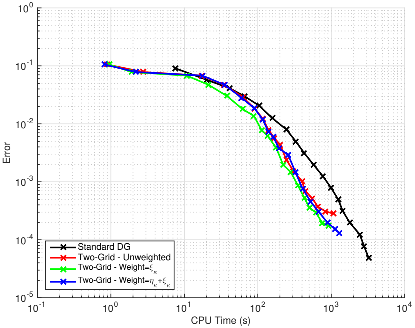
Error vs. #DoFs
(h-refinement)



Error vs. #DoFs
(hp-refinement)

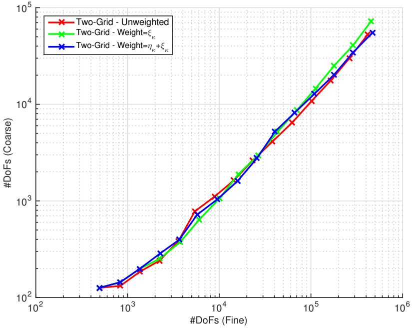


Error vs. Computation Time
(*h*-refinement)

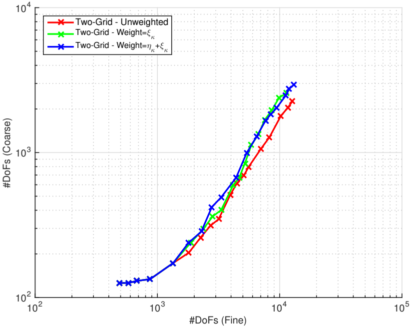


Error vs. Computation Time
(*hp*-refinement)

Quasilinear PDE: Singular Solution

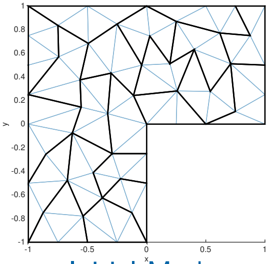


#DoFs: Fine vs. Coarse
(*h*-refinement)

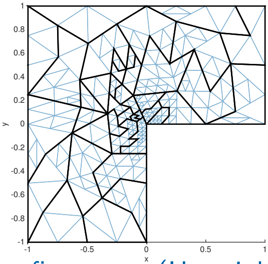


#DoFs: Fine vs. Coarse
(*hp*-refinement)

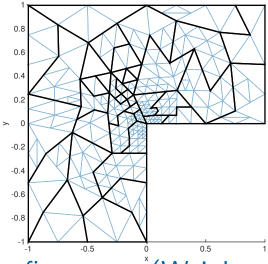
Quasilinear PDE: Singular Solution



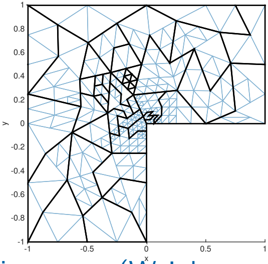
Initial Mesh



3 h-refinements (Unweighted)

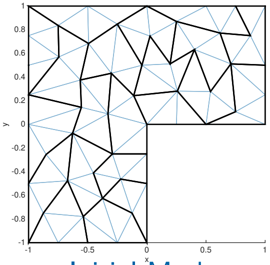


3 h-refinements (Weight $= \xi_\kappa$)

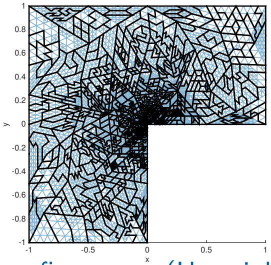


3 h-refinements (Weight $= \eta_\kappa + \xi_\kappa$)

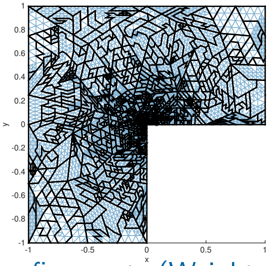
Quasilinear PDE: Singular Solution



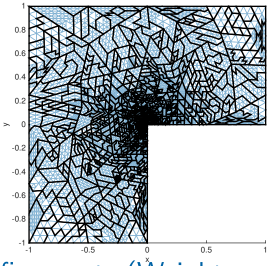
Initial Mesh



10 h-refinements (Unweighted)

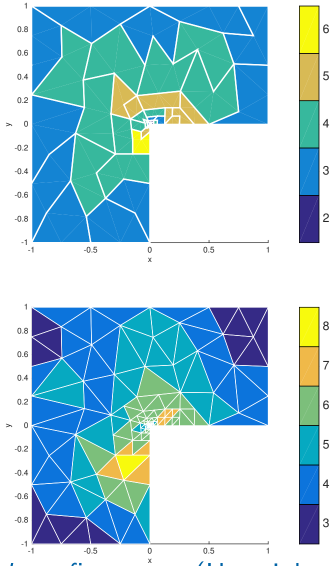


10 h-refinements (Weight $= \xi_\kappa$)

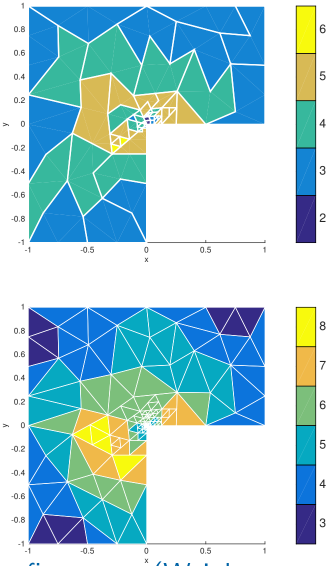


10 h-refinements (Weight $= \eta_\kappa + \xi_\kappa$)

Quasilinear PDE: Singular Solution



10 *hp*-refinements (Unweighted)



10 *hp*-refinements (Weight $= \eta_\kappa + \xi_\kappa$)

Non-Newtonian Fluid Problem

Given $\Omega \subset \mathbb{R}^d, d = 2, 3$ and $\boldsymbol{f} \in [L^2(\Omega)]^d$, find $(\boldsymbol{u}, p)$ such that

$$\begin{aligned} -\nabla \cdot \{\mu(\boldsymbol{x}, |\underline{e}(\boldsymbol{u})|)\underline{e}(\boldsymbol{u})\} + \nabla p &= \boldsymbol{f} && \text{in } \Omega, \\ \nabla \cdot \boldsymbol{u} &= 0 && \text{in } \Omega, \\ \boldsymbol{u} &= \boldsymbol{0} && \text{on } \Gamma, \end{aligned}$$

where $\underline{e}(\boldsymbol{u})$ is the *symmetric $d \times d$ strain tensor* defined by

$$e_{ij}(\boldsymbol{u}) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right).$$

Assumption

$\mu \in C(\bar{\Omega} \times [0, \infty))$ *and there exists positive constants* m_μ *and* M_μ *s.t.*

$$M_\mu(t - s) \leq \mu(\boldsymbol{x}, t)t - \mu(\boldsymbol{x}, s)s \leq M_\mu(t - s), \quad t \geq s \geq 0, \quad \boldsymbol{x} \in \bar{\Omega}.$$

hp-DGFEM

- Fine *hp*-DG finite element spaces:

$$\boldsymbol{V}_{hk}(\mathcal{T}_h, \boldsymbol{k}) = \{\boldsymbol{v} \in L^2(\Omega)^d : \boldsymbol{v}|_{\kappa_H} \circ T_\kappa \in \left[\mathcal{P}_{k_\kappa}(\hat{\kappa})\right]^d, \kappa \in \mathcal{T}_h\},$$
$$Q_{hk}(\mathcal{T}_h, \boldsymbol{k}) = \{q \in L^2_0(\Omega) : q|_{\kappa_H} \circ T_\kappa \in \mathcal{P}_{k_\kappa - 1}(\hat{\kappa}), \kappa \in \mathcal{T}_h\}.$$

- Fine *hp*-DG finite element spaces:

$$\boldsymbol{V}_{hk}(\mathcal{T}_h, \boldsymbol{k}) = \{\boldsymbol{v} \in L^2(\Omega)^d : \boldsymbol{v}|_{\kappa_H} \circ T_\kappa \in \left[\mathcal{P}_{k_\kappa}(\hat{\kappa})\right]^d, \kappa \in \mathcal{T}_h\},$$
$$Q_{hk}(\mathcal{T}_h, \boldsymbol{k}) = \{q \in L^2_0(\Omega) : q|_{\kappa_H} \circ T_\kappa \in \mathcal{P}_{k_\kappa - 1}(\hat{\kappa}), \kappa \in \mathcal{T}_h\}.$$

- Coarse *hp*-DG finite element spaces:

$$\boldsymbol{V}_{hk}(\mathcal{T}_h, \boldsymbol{k}) = \{\boldsymbol{v} \in L^2(\Omega)^d : \boldsymbol{v}|_{\kappa} \in \left[\mathcal{P}_{K_\kappa}(\kappa)\right]^d, \kappa \in \mathcal{T}_H\},$$
$$Q_{hk}(\mathcal{T}_h, \boldsymbol{k}) = \{q \in L^2_0(\Omega) : q|_{\kappa} \in \mathcal{P}_{K_\kappa - 1}(\kappa), \kappa \in \mathcal{T}_H\}.$$

- Fine *hp*-DG finite element spaces:

$$\boldsymbol{V}_{hk}(\mathcal{T}_h, \boldsymbol{k}) = \{\boldsymbol{v} \in L^2(\Omega)^d : \boldsymbol{v}|_{\kappa_H} \circ T_\kappa \in \left[\mathcal{P}_{k_\kappa}(\hat{\kappa})\right]^d, \kappa \in \mathcal{T}_h\},$$
$$Q_{hk}(\mathcal{T}_h, \boldsymbol{k}) = \{q \in L_0^2(\Omega) : q|_{\kappa_H} \circ T_\kappa \in \mathcal{P}_{k_\kappa - 1}(\hat{\kappa}), \kappa \in \mathcal{T}_h\}.$$

- Coarse *hp*-DG finite element spaces:

$$\boldsymbol{V}_{hk}(\mathcal{T}_h, \boldsymbol{k}) = \{\boldsymbol{v} \in L^2(\Omega)^d : \boldsymbol{v}|_{\kappa} \in \left[\mathcal{P}_{K_\kappa}(\kappa)\right]^d, \kappa \in \mathcal{T}_H\},$$
$$Q_{hk}(\mathcal{T}_h, \boldsymbol{k}) = \{q \in L_0^2(\Omega) : q|_{\kappa} \in \mathcal{P}_{K_\kappa - 1}(\kappa), \kappa \in \mathcal{T}_H\}.$$

- Jump operator: $\underline{[\![\boldsymbol{v}]\!]} = \boldsymbol{v}^+ \otimes \boldsymbol{n}^+ + \boldsymbol{v}^- \otimes \boldsymbol{n}^-$

(Standard) Interior Penalty Method

Find $(\boldsymbol{u}_{hk}, p_{hk}) \in \boldsymbol{V}_{hk}(\mathcal{T}_h, \boldsymbol{k}) \times Q_{hk}(\mathcal{T}_h, \boldsymbol{k})$ such that

$$\begin{aligned} A_{hk}(\boldsymbol{u}_{hk}; \boldsymbol{u}_{hk}, \boldsymbol{v}_{hk}) + B_{hk}(\boldsymbol{v}_{hk}, p_{hk}) &= F_{hk}(\boldsymbol{v}_{hk}) \\ -B_{hk}(\boldsymbol{u}_{hk}, q_{hk}) &= 0 \end{aligned}$$

for all $(\boldsymbol{v}_{hk}, q_{hk}) \in \boldsymbol{V}_{hk}(\mathcal{T}_h, \boldsymbol{k}) \times Q_{hk}(\mathcal{T}_h, \boldsymbol{k})$.

$$A_{hk}(\psi; \boldsymbol{u}, \boldsymbol{v}) = \int_\Omega \mu(|\underline{e}_h(\psi)|)\underline{e}_h(\boldsymbol{u}) : \underline{e}_h(\boldsymbol{v})\, d\boldsymbol{x}$$
$$- \int_{\mathcal{F}_h} \{\!\{\mu(|\underline{e}(\psi)|)\underline{e}(\boldsymbol{u})\}\!\} : \underline{[\![\boldsymbol{v}]\!]}\, ds$$
$$+ \theta \int_{\mathcal{F}_h} \{\!\{\mu(h_F^{-1}|\underline{[\![\psi]\!]}|)\underline{e}(\boldsymbol{v})\}\!\} : \underline{[\![\boldsymbol{u}]\!]}\, ds$$
$$+ \int_{\mathcal{F}_h} \sigma_{h,k}\underline{[\![\boldsymbol{u}]\!]} : \underline{[\![\boldsymbol{v}]\!]}\, ds,$$
$$B_{hk}(\boldsymbol{v}, q) = -\sum_{\kappa\in\mathcal{T}_h} \int_\kappa q\nabla\cdot\boldsymbol{v}\, d\boldsymbol{x} + \sum_{F\in\mathcal{F}_h} \int_F \{\!\{q\}\!\}[\![\boldsymbol{v}]\!]\, ds,$$
$$F_{hk}(v) = \sum_{\kappa\in\mathcal{T}_h} \int_\kappa \boldsymbol{f}\cdot\boldsymbol{v}\, d\boldsymbol{x}.$$

universität wien

Two-Grid Approximation

1. Construct $\boldsymbol{V}_{HK}(\mathcal{T}_H, \boldsymbol{K})$, $Q_{HK}(\mathcal{T}_H, \boldsymbol{K})$, $\boldsymbol{V}_{hk}(\mathcal{T}_h, \boldsymbol{k})$, and $Q_{hk}(\mathcal{T}_h, \boldsymbol{k})$.
2. Compute $(\boldsymbol{u}_{hk}, p_{HK}) \in \boldsymbol{V}_{HK}(\mathcal{T}_H, \boldsymbol{K}) \times Q_{HK}(\mathcal{T}_H, \boldsymbol{K})$ such that

$$\begin{aligned} A_{HK}(\boldsymbol{u}_{HK}; \boldsymbol{u}_{HK}, v_{HK}) + B_{HK}(\boldsymbol{v}_{HK}, p_{HK}) &= F_{HK}(\boldsymbol{v}_{HK}), \\ -B_{HK}(\boldsymbol{u}_{HK}, q_{HK}) &= 0 \end{aligned}$$

for all $(\boldsymbol{v}_{HK}, q_{HK}) \in \boldsymbol{V}_{HK}(\mathcal{T}_H, \boldsymbol{K}) \times Q_{HK}(\mathcal{T}_H, \boldsymbol{K})$.

Two-Grid Approximation

1. Construct $\boldsymbol{V}_{HK}(\mathcal{T}_H, \boldsymbol{K})$, $Q_{HK}(\mathcal{T}_H, \boldsymbol{K})$, $\boldsymbol{V}_{hk}(\mathcal{T}_h, \boldsymbol{k})$, and $Q_{hk}(\mathcal{T}_h, \boldsymbol{k})$.
2. Compute $(\boldsymbol{u}_{hk}, p_{HK}) \in \boldsymbol{V}_{HK}(\mathcal{T}_H, \boldsymbol{K}) \times Q_{HK}(\mathcal{T}_H, \boldsymbol{K})$ such that

$$\begin{aligned} A_{HK}(\boldsymbol{u}_{HK}; \boldsymbol{u}_{HK}, v_{HK}) + B_{HK}(\boldsymbol{v}_{HK}, p_{HK}) &= F_{HK}(\boldsymbol{v}_{HK}), \\ -B_{HK}(\boldsymbol{u}_{HK}, q_{HK}) &= 0 \end{aligned}$$

   for all $(\boldsymbol{v}_{HK}, q_{HK}) \in \boldsymbol{V}_{HK}(\mathcal{T}_H, \boldsymbol{K}) \times Q_{HK}(\mathcal{T}_H, \boldsymbol{K})$.
3. Determine $(\boldsymbol{u}_{2G}, p_{2G}) \in \boldsymbol{V}_{hk}(\mathcal{T}_h, \boldsymbol{k}) \times Q_{hk}(\mathcal{T}_h, \boldsymbol{k})$ such that

$$\begin{aligned} A_{hk}(u_{HK}; \boldsymbol{u}_{2G}, \boldsymbol{v}_{hk}) + B_{hk}(\boldsymbol{v}_{hk}, p_{2G}) &= F_{hk}(v_{hk}), \\ -B_{hk}(\boldsymbol{u}_{2G}, q_{hk}) &= 0 \end{aligned}$$

   for all $(\boldsymbol{v}_{hk}, q_{hk}) \in \boldsymbol{V}_{hk}(\mathcal{T}_h, \boldsymbol{k}) \times Q_{hk}(\mathcal{T}_h, \boldsymbol{k})$.

[C., & Houston 2014]

Lemma (Standard Non-Newtonian Fluid DGFEM)

The following bound holds:

$$\|(\boldsymbol{u}-\boldsymbol{u}_{hk}, p - p_{hk})\|^2_{DG} \leq C_3 \sum_{\kappa\in\mathcal{T}_h} \eta_\kappa^2 \qquad .$$

Here the local error indicators η_κ *are defined, for all* $\kappa \in \mathcal{T}_h$, *as*

$$\eta_\kappa^2 = h_\kappa^2 k_\kappa^{-2} \|\boldsymbol{f} + \nabla\cdot\{\mu(|\underline{e}(\boldsymbol{u}_{hk})|)\underline{e}(\boldsymbol{u}_{hk})\} - \nabla p_{hk}\|^2_{L^2(\kappa)} + \|\nabla\cdot\boldsymbol{u}_{hk}\|^2_{L^2(\kappa)}$$
$$+ h_\kappa k_\kappa^{-1} \|[\![p_{hk}]\!] - [\![\mu(|\underline{e}(\boldsymbol{u}_{hk})|)\underline{e}(\boldsymbol{u}_{hk})]\!]\|^2_{L^2(\partial\kappa\setminus\Gamma)} + \gamma^2 k_\kappa^3 h_\kappa^{-1} \|\underline{[\![\boldsymbol{u}_{hk}]\!]}\|^2_{L^2(\partial\kappa)}$$

Proof.

See C., Houston, Süli & Wihler 2013. □

Lemma (Two-Grid Non-Newtonian Fluid Approximation)

The following bound holds:

$$\|(\boldsymbol{u} - \boldsymbol{u}_{2G}, p - p_{2G})\|_{DG}^2 \leq C_4 \sum_{\kappa\in\mathcal{T}_h} \left(\eta_\kappa^2 + \xi_\kappa^2\right).$$

Here the local error indicators η_κ are defined, for all $\kappa \in \mathcal{T}_h$, as

$$\eta_\kappa^2 = h_\kappa^2 k_\kappa^{-2}\|\boldsymbol{f} + \nabla\cdot\{\mu(|\underline{e}(\boldsymbol{u}_{HK})|)\underline{e}(\boldsymbol{u}_{2G})\} - \nabla p_{2G}\|_{L^2(\kappa)}^2 + \|\nabla\cdot\boldsymbol{u}_{2G}\|_{L^2(\kappa)}^2$$
$$+ h_\kappa k_\kappa^{-1}\|[\![p_{2G}]\!] - [\![\mu(|\underline{e}(\boldsymbol{u}_{HK})|)\underline{e}(\boldsymbol{u}_{2G})]\!]\|_{L^2(\partial\kappa\setminus\Gamma)}^2 + \gamma^2 k_\kappa^3 h_\kappa^{-1}\|\underline{[\![\boldsymbol{u}_{2G}]\!]}\|_{L^2(\partial\kappa)}^2$$

and the local two-grid error indicators are defined, for all $\kappa \in \mathcal{T}_h$, as

$$\xi_\kappa^2 = \|(\mu(|\underline{e}(\boldsymbol{u}_{HK})|) - \mu(|\underline{e}(\boldsymbol{u}_{2G})|))\nabla\boldsymbol{u}_{2G}\|_{L^2(\kappa)}^2.$$

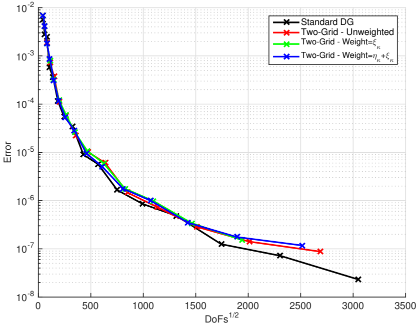
Proof.

See C., & Houston 2014. □

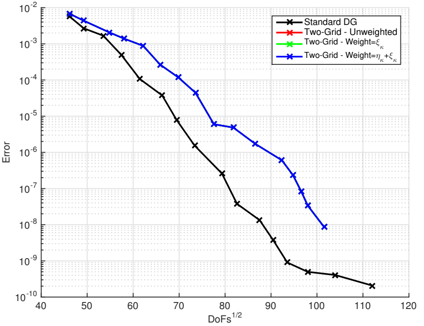
We let $\Omega = (-1,1)^2 \setminus [0,1) \times (-1,0], \mu(\boldsymbol{x}, |\underline{e}(u)|) = 2 + \frac{1}{1+|\underline{e}(\boldsymbol{u})|^2}$ and select f so that

$$\boldsymbol{u}(x,y) = \begin{pmatrix} -\mathrm{e}^x(y\cos y + \sin y) \\ e^x y \sin y \end{pmatrix}$$

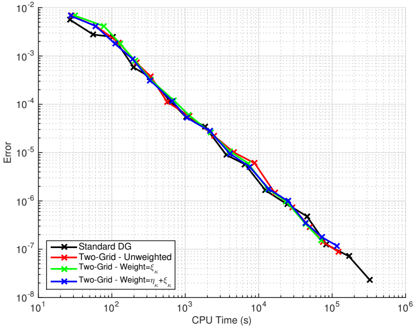
$$p(x,y) = 2\mathrm{e}^x \sin y - \frac{2}{3}(1-e)(cos1 - 1).$$



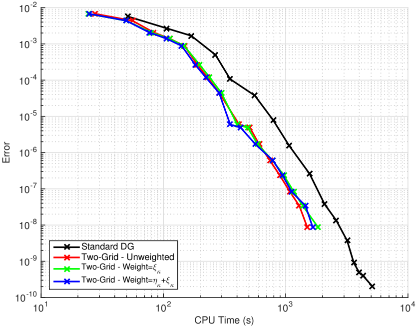
Error vs. #DoFs
(h-refinement)



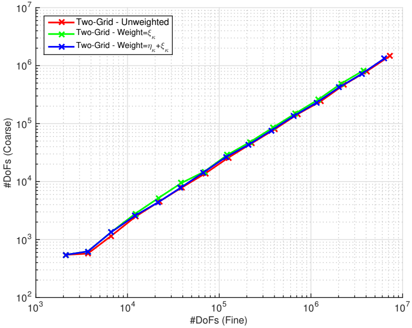
Error vs. #DoFs
(hp-refinement)



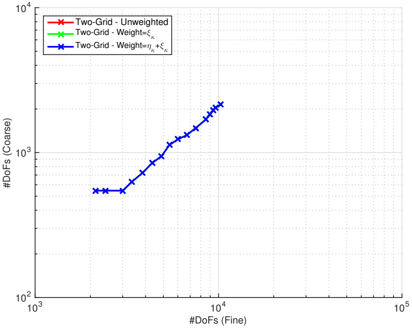
Error vs. Computation Time
(*h*-refinement)



Error vs. Computation Time
(*hp*-refinement)

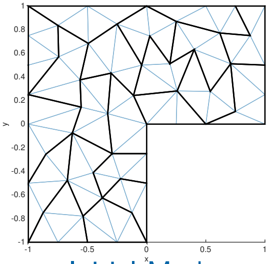


#DoFs: Fine vs. Coarse
(*h*-refinement)

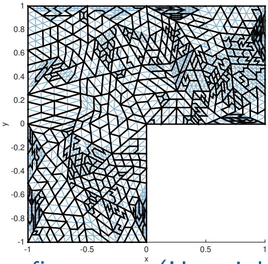


#DoFs: Fine vs. Coarse
(*hp*-refinement)

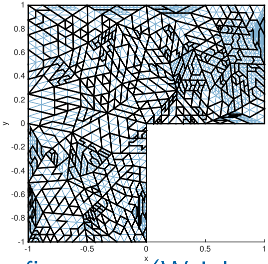
Non-Newtonian Fluid Flow: Smooth Solution



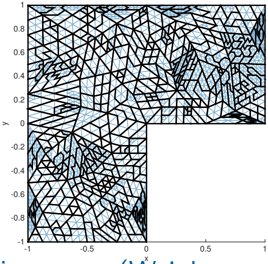
Initial Mesh



6 h-refinements (Unweighted)

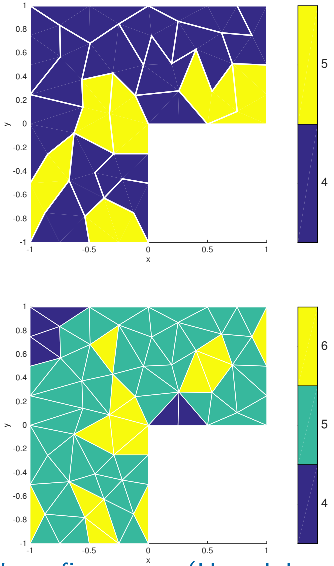


6 h-refinements (Weight $= \xi_\kappa$)

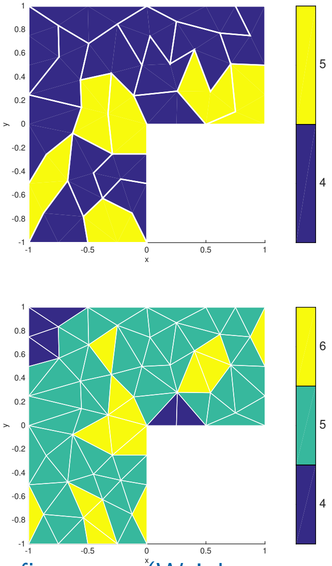


6 h-refinements (Weight $= \eta_\kappa + \xi_\kappa$)

Non-Newtonian Fluid Flow: Smooth Solution



6 *hp*-refinements (Unweighted)



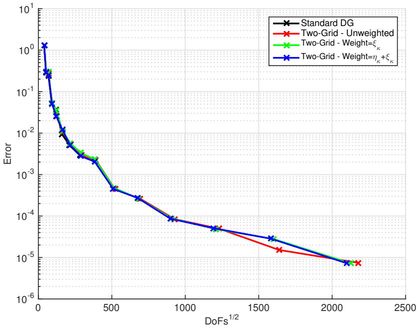
6 *hp*-refinements (Weight $= \eta_\kappa + \xi_\kappa$)

We let $\Omega = (0,1)^2 \setminus [0,1) \times (-1,0]$, with Carreau Law nonlinearity $\mu(\boldsymbol{x}, |\underline{e}(u)|) = k_\infty + (k_0 - k_\infty)(1 + \lambda |\underline{e}(\boldsymbol{u})|^2)^{(\vartheta-2)/2}$, and select f so that

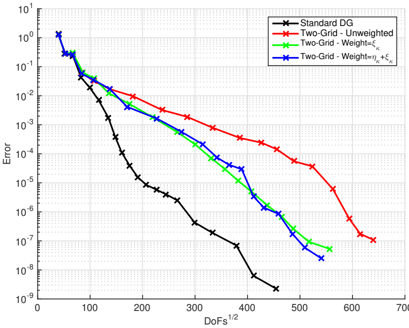
$$\boldsymbol{u}(x,y) = \begin{pmatrix} \left(1 - \cos\left(2\frac{\pi(e^{\vartheta x}-1)}{e^{\vartheta}-1}\right)\right) \sin(2\pi y) \\ -\vartheta e^{\vartheta x} \sin\left(2\frac{\pi(e^{\vartheta x}-1)}{e^{\vartheta}-1}\right) \frac{1-\cos(2\pi y)}{e^{\vartheta}-1} \end{pmatrix}$$

$$p(x,y) = 2\pi\vartheta e^{\vartheta x} \sin\left(2\frac{\pi(e^{\vartheta x}-1)}{e^{\vartheta}-1}\right) \frac{\sin(2\pi y)}{e^{\vartheta}-1}.$$

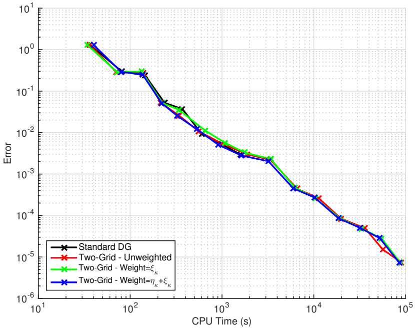
We select $k_\infty = 1$, $k_0 = 2$, $\lambda = 1$, and $\vartheta = 1.2$.



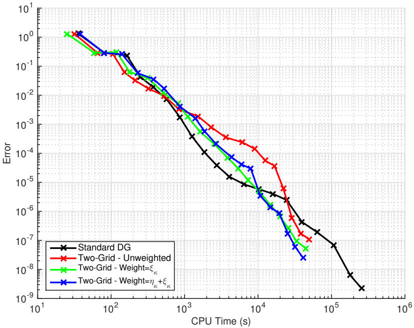
Error vs. #DoFs
(h-refinement)



Error vs. #DoFs
(hp-refinement)

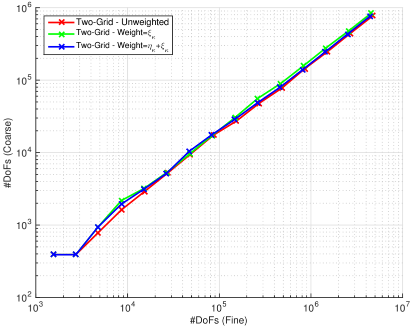


Error vs. Computation Time (*h*-refinement)

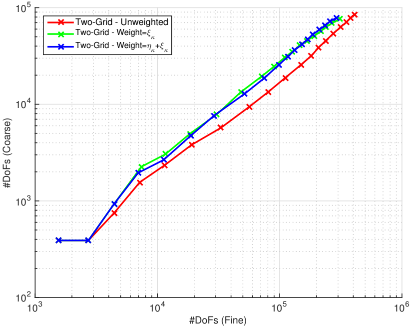


Error vs. Computation Time (*hp*-refinement)

Non-Newtonian Fluid Flow: Cavity Problem

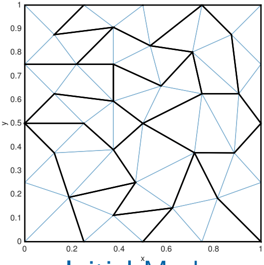


#DoFs: Fine vs. Coarse
(*h*-refinement)

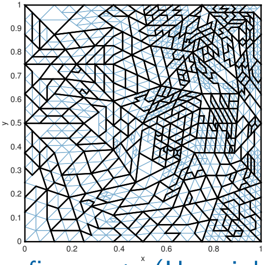


#DoFs: Fine vs. Coarse
(*hp*-refinement)

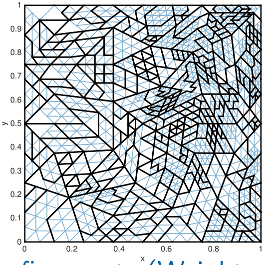
Non-Newtonian Fluid Flow: Cavity Problem



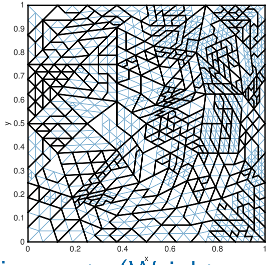
Initial Mesh



6 h-refinements (Unweighted)



6 h-refinements (Weight $= \xi_\kappa$)



6 h-refinements (Weight $= \eta_\kappa + \xi_\kappa$)

Non-Newtonian Fluid Flow: Cavity Problem

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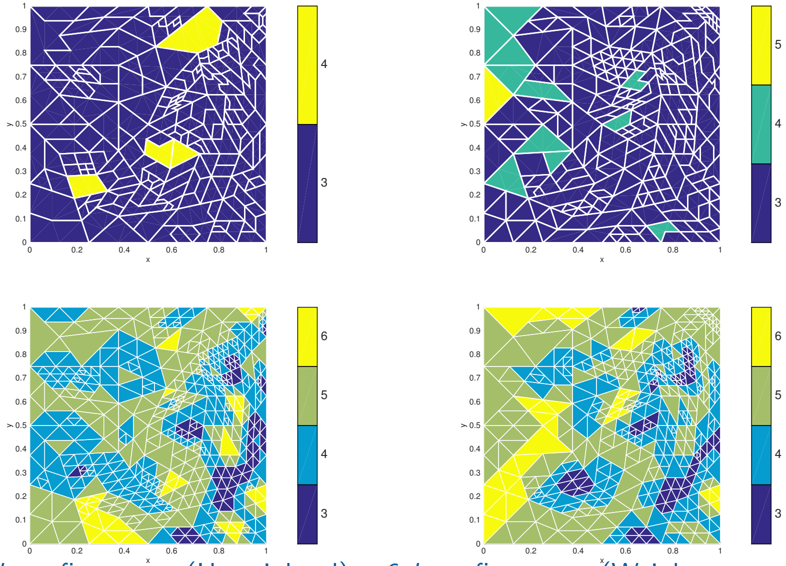

6 *hp*-refinements (Unweighted)

6 *hp*-refinements (Weight $= \eta_\kappa + \xi_\kappa$)

Summary:

- Two-Grid DG *a posteriori* error estimates still hold for agglomerated coarse mesh of polygons and fine mesh of simplices.
- We can adaptively refine the coarse mesh based on the error estimates.

Future Aims:

- *A priori* error bounds.
- Extend to general nonlinearities.