Implementation of FEM

Consider the discrete problem: Find $u_h \in V_h$ s.t.

$$a(u_h, v_h) = (f, v_h) \quad \forall v_h \in V_h$$

where a is defined as integrals over Ω; e.g.

$$a(u,v) = \int_\Omega \nabla u \cdot \nabla v \, dx + \int_\Omega c u v \, dx$$

We can write this as

$$a(u,v) = \sum_{T \in \mathcal{T}_h} a_T(u,v)$$

where $a_T(u,v) = \int_T \nabla u \cdot \nabla v \, dx + \int_T c u v \, dx$

Let $\varphi_1, \dots, \varphi_N$ be the basis of V_h; then, $u_h = \sum_{j=1}^N u_j \varphi_j$

and discrete problem is equivalent to algebraic problem

$$\sum_{j=1}^N a(\varphi_j, \varphi_i) u_j = (f, \varphi_i) \quad i = 1, \dots, N;$$

i.e, to a linear system

$$AU = F$$

where $a_{ij} = a(\varphi_j, \varphi_i)$, $F_i = (f, \varphi_i)$, $U = (u_1, \dots u_n)$.

To compute matrix entries we can use following algorithm:

$a_{ij} = 0 \quad \forall i,j$
for $T \in \mathcal{T}_h$ do
  $a_{ij} := a_{ij} + a_T(\varphi_j, \varphi_i) \quad \forall i,j$
end for

For standard basis functions

$$a_T(\varphi_j, \varphi_i) \neq 0 \iff i,j \in I_h(T)$$

where $I_h(T)$ is set of nodes in T.

Therefore, we have the algorithm

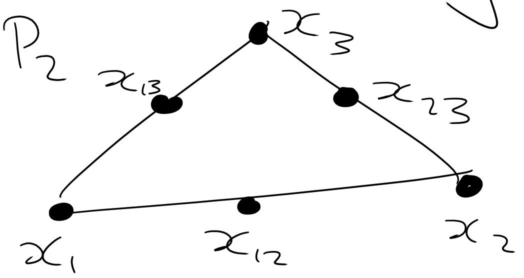
$$
\begin{aligned}
&a_{ij} = 0 \quad \forall i,j \\
&\text{for } T \in \mathcal{T}_h \text{ do} \\
&\quad \text{for } i,j \in I_h(T) \text{ do} \\
&\qquad a_{ij} := a_{ij} + a_T(\varphi_j, \varphi_i)
\end{aligned}
$$

The matrix $(a_T(\varphi_j,\varphi_i))_{i,j\in I_h(T)}$ is called the local stiffness matrix

- General way to compute $a_T(\varphi_j,\varphi_i)$ is to transform integrals to reference element, applying numerical integration on the reference element.

- For simplices we can use formula for barycentric coordinates directly:

P_2



$$\int_T \nabla u \cdot \nabla v \, dx$$

$$\varphi_i = \lambda_i(2\lambda_i - 1) \qquad \varphi_{ij} = 4\lambda_i\lambda_j$$

$$\varphi_1 = \lambda_1(2\lambda_1 - 1) = 2\lambda_1^2 - \lambda_1 \qquad \lambda_{12} = 4\lambda_1\lambda_2$$

$$\int_T \nabla\varphi_1 \cdot \nabla\varphi_{12}\, dx = \int_T (4\lambda_1 - 1)\nabla\lambda_1 \left[4\lambda_1\nabla\lambda_2 + 4\lambda_2\nabla\lambda_1\right] dx$$

$$= 4\int_T (4\lambda_1^2 - \lambda_1)\underbrace{\nabla\lambda_1\cdot\nabla\lambda_2}_{=\text{const}}\, dx + 4\int_T (4\lambda_1\lambda_2 - \lambda_2)\underbrace{\nabla\lambda_1\cdot\nabla\lambda_1}_{=\text{const}}\, dx$$

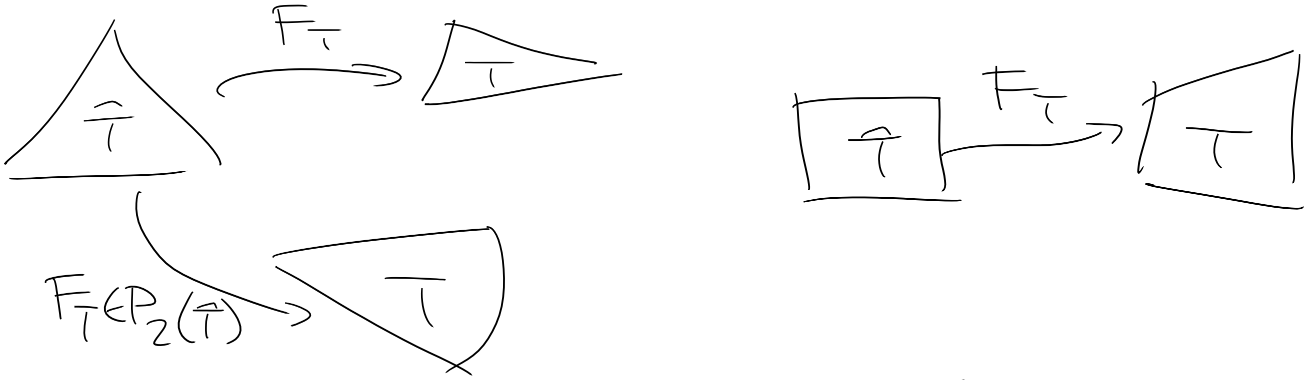
Additionally use identity that

$$\int_T \lambda_1^{\alpha_1}\cdots\lambda_{n+1}^{\alpha_{n+1}}\, dx = \frac{\alpha_1!\cdots\alpha_{n+1}!\, n!}{(\alpha_1 + \cdots + \alpha_{n+1} + n)!}|T|$$

$$\int_T \lambda_1 \lambda_2 \, dx = \frac{1\cdot 1\cdot 2}{(1+1+2)!}|T| = \frac{|T|}{2}$$

$$\int_T \lambda_1^2 \, dx = \frac{2\cdot 2}{4!}|T| = \frac{|T|}{6}$$

Integrals on reference element



Mapping does not have to be affine.

u on $T \to \hat{u} = u \circ F_T \qquad \hat{u}(\hat{x}) = u(F_T(\hat{x}))$

$$\frac{\partial \hat{u}}{\partial \hat{x}_i}(\hat{x}) = \sum_{j=1}^{n} \frac{\partial u}{\partial x_j}(F_T(\hat{x})) \frac{\partial F_T(\hat{x})_j}{\partial \hat{x}_i}$$

Jacobian of F_T:

$$DF_T(\hat{x}) = \left(\frac{\partial F_T(\hat{x})_i}{\partial \hat{x}_j} \right)_{i,j=1}^{n}$$

$$\Rightarrow \hat{\nabla} \hat{u}(\hat{x}) = DF_T(\hat{x})^T \nabla u(F_T(\hat{x}))$$

$$\int_T \nabla u \cdot \nabla v \, dx = \int_{F_T^{-1}(T)} (\nabla u)(F_T(\hat{x})) \cdot (\nabla v)(F_T(\hat{x})) |\det DF_T(\hat{x})| \, dx$$

$$= \int_T DF_T(\hat{x})^{-T} \hat{\nabla} \hat{u}(\hat{x}) \cdot DF_T(\hat{x})^{-T} \hat{\nabla} \hat{u}(\hat{x}) |\det DF_T(\hat{x})| \, dx$$

If F_T is an invertible affine mapping then

$$DF_T = B_T \quad \text{where } F_T(\hat{x}) = B_T \hat{x} + b_T$$

Thus, $\int_T \nabla u \cdot \nabla v \, dx = \int_{\hat{T}} B_T^{-T} \hat{\nabla} \hat{u} \cdot B_T^{-T} \hat{\nabla} \hat{v} \, |\det B_T| \, d\hat{x}$

Then, the integral can be computed by numerical integration.

Quadrature formulas

$$\int_T \varphi \, dx \approx \sum_{l=1}^{L} \omega_{l,T} \, \varphi(b_{l,T})$$

weights nodes

Examples:

$-1D$: $T = [a_1, a_2]$ $\varphi_i = \varphi(a_i)$ $a_{12} = \frac{a_1 + a_2}{2}$

Midpoint rule:

$$\int_T \varphi \, dx \approx |T| \varphi_{12}$$ – exact for $\forall \varphi \in P_1(T)$

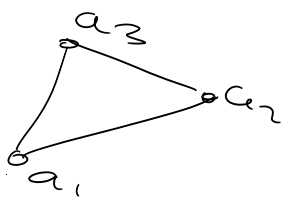
Trapezoidal rule:

$$\int_T \varphi \, dx \approx \frac{|T|}{2}(\varphi_1 + \varphi_2)$$ – exact for $\forall \varphi \in P_1(T)$

Simpson's rule:

$$\int_T \varphi \, dx \approx \frac{|T|}{6}(\varphi_1 + 4\varphi_{12} + \varphi_2)$$ – exact for $\forall \varphi \in P_3(T)$

$-2D$



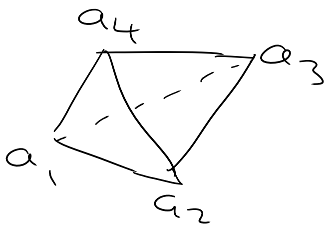
$a_{123} = \frac{1}{3}(a_1 + a_2 + a_3)$

- $\int_T \varphi(x) \, dx \approx |T| \varphi_{123}$ – exact for $\forall \varphi \in P_1(T)$
- $\int_T \varphi(x) \, dx \approx \frac{|T|}{3}(\varphi_1 + \varphi_2 + \varphi_3)$ – exact for $\forall \varphi \in P_1(T)$
- $\int_T \varphi(x) \, dx \approx \frac{|T|}{3}(\varphi_{12} + \varphi_{23} + \varphi_{13})$ – exact for $\forall \varphi \in P_2(T)$

- $\int_T \varphi(x)\,dx \approx \frac{|T|}{60}\Big[3(\varphi_1+\varphi_2+\varphi_3)+8(\varphi_{12}+\varphi_{13}+\varphi_{23})$
$+27\,\varphi_{123}\Big]$

— exact $\forall \varphi \in P_3(T)$

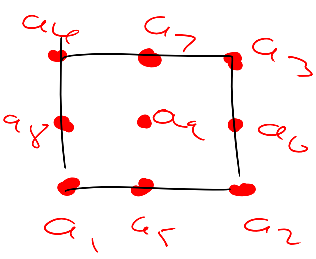
- 3D



$a_{1234} = \frac{1}{4}(a_1+a_2+a_3+a_4)$

- $\int_T \varphi\,dx \approx |T|\,\varphi_{1234}$ — exact $\forall \varphi \in P_1(T)$
- $\int_T \varphi\,dx \approx \frac{|T|}{4}(\varphi_1+\varphi_2+\varphi_3+\varphi_4)$ — exact $\forall \varphi \in P_1(T)$
- $\int_T \varphi\,dx \approx \frac{|T|}{20}(\varphi_1+\varphi_2+\varphi_3+\varphi_4+16\,\varphi_{1234})$ — exact $\forall \varphi \in P_2(T)$

- 2D - square $[0,1]^2$
  - $\int_T \varphi(x)\,dx \approx \frac{1}{4}\sum_{i=1}^{4}\varphi_i$ — exact $\varphi \in Q_1(T)$
  - $\int_T \varphi(x)\,dx \approx \frac{1}{36}\Big[\sum_{i=1}^{4}\varphi_i + 4\sum_{i=5}^{8}\varphi_i + 16\,\varphi_9\Big]$ — exact $\varphi \in Q_3(T)$



Gauss integration formulas — Efficient based on special choose of nodes
(Gauss-Quadrature)

- 1D: $T = [-1,1]$ $\quad [0,1]$
  - $\int_T \varphi(x)\,dx \approx \frac{1}{2}\big(\varphi(n_1)+\varphi(n_2)\big)$

where $x_1 = -\frac{1}{\sqrt{3}} \;\Big|\; \frac{1}{2} - \frac{\sqrt{3}}{6}$ $\quad x_2 = \frac{1}{\sqrt{3}} \;\Big/\; \frac{1}{2} + \frac{\sqrt{3}}{6}$

Exact for $\forall \varphi \in P_3(T)$

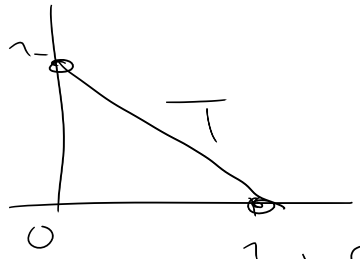
- $\int_T \varphi(x)dx \approx \frac{1}{9} \Big| \frac{1}{18} \left[5\varphi(\xi_1) + 8\varphi(\xi_2) + 5\varphi(\xi_3)\right]$

where $\xi_1 = -\sqrt{\frac{3}{5}}$, $\xi_2 = 0$, $\xi_3 = \sqrt{\frac{3}{5}}$

$\left(\frac{1}{2} - \frac{1}{10}\sqrt{15}\right)$ $\quad \left(\frac{1}{2}\right)$ $\quad \left(\frac{1}{2} + \frac{1}{10}\sqrt{15}\right)$

Exact for $\forall \varphi \in P_5(T)$

- 2D



$$\int_T \varphi(x)dx \approx \frac{1}{6}\left[\varphi\left(\tfrac{1}{6}, \tfrac{1}{6}\right) + \varphi\left(\tfrac{2}{3}, \tfrac{1}{6}\right) + \varphi\left(\tfrac{1}{6}, \tfrac{2}{3}\right)\right]$$

Exact for $\forall \varphi \in P_2(T)$

Consider Barycentric coordinates $\lambda_1 = x_1$, $\lambda_2 = x_2$, $\lambda_3 = 1 - x_1 - x_2$

$\left(\frac{1}{6}, \frac{1}{6}\right) \Rightarrow \lambda_1 = \frac{1}{6}, \lambda_2 = \frac{1}{6}, \lambda_3 = \frac{2}{3}$

$\left(\frac{2}{3}, \frac{1}{6}\right) \Rightarrow \lambda_1 = \frac{2}{3}, \lambda_2 = \frac{1}{6}, \lambda_3 = \frac{1}{6}$

$\left(\frac{1}{6}, \frac{2}{3}\right) \Rightarrow \lambda_1 = \frac{1}{6}, \lambda_2 = \frac{2}{3}, \lambda_3 = \frac{1}{6}$

Symmetric in sense of permutations of $\frac{1}{6}, \frac{1}{6}, \frac{2}{3}$ for barycentric coordinates

$\hookrightarrow$ Dunavant, D.A, High degree efficient symmetrical Gaussian quadrature rules for the triangle. Int. J. Numer. Meth. Engng. 21: 1129–1148 (1985)

$\rightarrow$ Appendix II list barycentric coordinates & weights.

Note, that weights are for $|T| = 1$, but $|T| = \frac{1}{2} \Rightarrow$ divide weights by 2.

- 2D - square $[-1,1]^2$

$\eta_1 = \frac{-1}{\sqrt{3}}, \eta_2 = \frac{1}{\sqrt{3}}, \xi_1 = -\sqrt{\frac{3}{5}}, \xi_2 = 0, \xi_3 = \sqrt{\frac{3}{5}}$

- $\int_T \varphi(x)dx \approx \sum_{i,j=1}^{2} \varphi(\eta_i, \eta_j)$ - exact $\forall \varphi \in Q_3(T)$
- $\int_T \varphi(x)dx \approx \sum_{i,j=1}^{3} \omega_i \omega_j \varphi(\xi_i, \xi_j)$ - exact $\forall \varphi \in Q_5(T)$

where $\omega_1 = \omega_3 = \frac{5}{9}, \omega_2 = \frac{8}{9}$

$\hookrightarrow$ "Cartesian product" of 1D quadrature rules, where $\omega_i, i=1,..,3$ are the 1D quadrature weights

- More generally
  $$\int_T \varphi(x)dx \approx \sum_{i,j=1}^{m} \omega_i \omega_j \varphi(x_i, x_j)$$
  where x_i, ω_i are the m 1D-quadrature points & weights.

Note on derivation of 1D points

The 1D points and weights are sometimes called the Gauss-Legendre quadrature points as on the interval $[-1,1]$, given the Legendre polynomials $P_m(x)$ (where $P_m(1)=1$), the i-th Gauss Legendre node $x_i, i=1,..,m$ is given by the ith root of P_m & weights by formula

$$\omega_i = \frac{2}{(1-x_i)^2 [P_n'(x_i^2)]^2}, i=1,...,m$$

Furthermore, $x_{m+1-i} = -x_i$ for $i=1,...,\lfloor \frac{m}{2} \rfloor$

and if i is odd $x_{(\frac{i}{2}+1)} = 0$