Error Estimates for non-conforming FE discretisations

Conforming error estimates used Cea's lemma; however, here not subspaces of variational form:

$-\Delta u = f$ in Ω, $u = 0$ on $\partial\Omega$, $\Omega \subset \mathbb{R}$ polygonal

$\{\mathcal{T}_h\}$ family of regular triangulations of Ω consisting of triangles or quads. satisfying $(\mathcal{T}_{h,1})-(\mathcal{T}_{h,5})$,

$\forall T \in \mathcal{T}_h$: (T, P_T, Σ_T) affine-equivalent to $(\hat{T}, \hat{P}, \hat{\Sigma})$

Assume $\mathbb{P}_k(T) \subset P_T \subset H^1(T)$ $\forall T \in \mathcal{T}_h$ (with some $k \geq 1$)

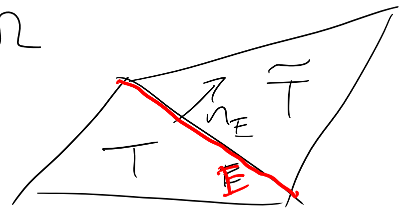
FE spaces:

$$V_h = \{ v_h \in L^2(\Omega) : v_h|_T \in P_T \ \forall T \in \mathcal{T}_h, \int_E [\![v_h]\!]_E \, q \, ds = 0 \ \forall q \in P_{k-1}(E) \ \forall E \in \mathcal{E}_h \}$$

$\mathcal{E}_h$ - set of edges of $\mathcal{T}_h$, n_E - unit normal to E.

$[\![v_h]\!]_E = (v_h|_T)|_E - (v_h|_{\tilde{T}})|_E$ for $E \not\subset \partial\Omega$

$[\![v_h]\!]_E = v_h|_E$ for $E \subset \partial\Omega$



$V_h \sim H_0^1(\Omega)$ but $V_h \not\subset H^1(\Omega)$

Weak formulation: $u \in H_0^1(\Omega)$: $a(u,v) = (f,v)$ $\forall v \in H_0^1(\Omega)$

$$a(u,v) = \int_\Omega \nabla u \cdot \nabla v \, dx$$

Discrete problem: $u_h \in V_h$: $a_h(u_h, v_h) = (f, v_h)$ $\forall v_h \in V_h$

$$a_h(u,v) = \sum_{T \in \mathcal{T}_h} \int_T \nabla u \cdot \nabla v \, dx$$

$$|v|_{1,h} = \Big[\sum_{T \in \mathcal{T}_h} |v|_{1,T}^2 \Big]^{1/2}$$ is a norm in V_h.

Clearly $|v|_{1,h}$ is a semi-norm.

Let $|v_h|_{1,h} = 0$ for some $v_h \in V_h$. Then, $|v_h|_{1,T} = 0 \;\; \forall T \in V_h$

$\Rightarrow \nabla v_h|_T = 0 \;\; \forall T \in \mathcal{T}_h \Rightarrow v_h|_T = \text{const} \;\; \forall T \in \mathcal{T}_h$.

Consider any interior edge $E \in \mathcal{E}_h$. Then,

$$\int_E \underbrace{[\![v_h]\!]_E}_{\text{const}} \, ds = 0 \Rightarrow [\![v_h]\!]_E = 0 \Rightarrow v_h|_T = v_h|_{\hat{T}} \text{ for } T, \hat{T} \supset E$$

$\Rightarrow v_h = \text{const.}$ in Ω.

Consider any $E \subset \partial\Omega$ Then $0 = \int_E [\![v_h]\!]_E \, ds = \int_E v_h \, ds = v_h |E|$

$\Rightarrow v_h = 0$

$\Rightarrow v_h = 0$ on $\partial\Omega$, constant on T, $\forall T \in \mathcal{T}_h$ & on $E \in \mathcal{E}_h$

$\Rightarrow v_h = 0$ on whole space

$\Rightarrow |\cdot|_{1,h}$ is norm of V_h.

Recall conforming case:

AVP: $u \in V \quad a(u,v) = \langle f, v \rangle \quad \forall v \in V$

Disc.: $u_h \in V_h \quad a(u_h, v_h) = \langle f, v \rangle \quad \forall v_h \in V_h$

$V_h \subset V$: $\quad a(u, v_h) = \langle f, v_h \rangle \quad \forall v_h \in V_h$ (consistency)

Let's assume $u \in H^2(\Omega)$.

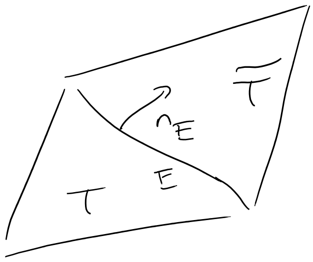
$\forall v \in C_0^\infty(\Omega)$: $\quad (f, v) = a(u,v) = \int_\Omega \nabla u \cdot \nabla v \, dx = -\int_\Omega \Delta u \, v \, dx$

$\Rightarrow -\Delta u = f$ a.e. in Ω.

$$-\int_T \Delta u \cdot v_h = \int_T f v_h \qquad \forall v_h \in V_h, \; T \in \mathcal{T}_h$$

$$\sum_{T \in \mathcal{T}_h} \int_T \nabla u \cdot \nabla v_h \, dx - \sum_T \int_{\partial T} \frac{\partial u}{\partial n_T} v_h \, ds = \sum_T \int_T f v_h \, dx$$

$$\Rightarrow a_h(u,v_h) = (f,v_h) + \sum_{T\in\mathcal{T}_h} \int_{\partial T} \frac{\partial u}{\partial n_T} v_h \, ds$$



$$\int_E \frac{\partial u}{\partial n_T} v_h|_T \, ds + \int_E \frac{\partial u}{\partial n_{\tilde{T}}} v_h|_{\tilde{T}} \, ds$$

$$= \int_E \frac{\partial u}{\partial n_E} v_h|_T \, ds - \int_E \frac{\partial u}{\partial n_E} v_h|_{\tilde{T}} \, ds$$

$$= \int_E \frac{\partial u}{\partial n_E} [\![v_h]\!] \, ds$$

$E \subset \partial\Omega$ $\quad \int_E \frac{\partial u}{\partial n_T} v_h \, ds = \int_E \frac{\partial u}{\partial n_E} [\![v_h]\!]_E \, ds$

$$\Rightarrow a_h(u,v_h) = \underbrace{(f,v_h)}_{a_h(u_h,v_h)} + \underbrace{\sum_{E\in\mathcal{E}_h} \int_E \frac{\partial u}{\partial n_E} [\![v_h]\!]_E \, ds}_{\text{consistency error}} \quad \forall v_h \in V_h$$

$$\Rightarrow a_h(u-u_h,v_h) = \sum_{E\in\mathcal{E}_h} \int_E \frac{\partial u}{\partial n_E} [\![v_h]\!]_E \, ds$$

Consider any $v_h, w_h \in V_h$. Then,

$$\sup_{w_h\in V_h} \frac{a_h(v_h-u_h,w_h)}{|w_h|_{1,h}} \le \sup_{w_h\in V_h} \frac{a_h(v_h-u,w_h)}{|w_h|_{1,h}} + \sup \frac{a_h(u-u_h,w_h)}{|w_h|_{1,h}}$$

$$|u_h-v_h|_{1,h} + |u-v_h|_{1,h} \le 2|u-v_h|_{1,h} + \sup_{w_h\in V_h} \frac{a_h(u-u_h,w_h)}{|w_h|_{1,h}}$$

$$\Rightarrow |u-u_h|_{1,h} \le 2|u-v_h|_{1,h} + \sup \frac{a_h(u-u_h,w_h)}{|w_h|_{1,h}} \quad \forall v_h \in V_h$$

$$\le 2 \inf_{v_h\in V_h} |u-v_h|_{1,h} + \sup_{w_h\in V_h} \frac{\sum_{E\in\mathcal{E}_h} \int_E \frac{\partial u}{\partial n_E} [\![w_h]\!]_E \, ds}{|w_h|_{1,h}}$$

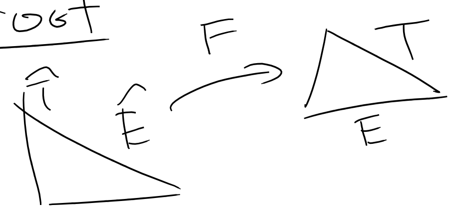
For any edge E, we define the orthogonal projection $\mu_E^k : L^2(E) \xrightarrow{\text{orth}} P_k(E)$;

i.e., $\displaystyle\int_E (v - \mu_E^k v) q \, ds = 0 \quad \forall q \in P_k(E), \forall v \in L^2(E)$.

<u>Lemma</u> $\exists C > 0$ such that for all $T \in \mathcal{T}_h, E \subset T$, $v \in H^1(T)$ & $z \in H^{k+1}(\Omega)$

$$\left| \int_E v (z - \mu_E^k z) \, ds \right| \le C h_E^{k+1} |v|_{1,T} |z|_{k+1,T}$$

<u>Proof</u>



$F(\hat{T}) = T, \; F(\hat{E}) = E$

$\displaystyle\int_E v \, ds = \int_{\hat{E}} \hat{v} \omega \, d\hat{s}$ $\quad |E| = |\hat{E}| \cdot \omega$

$\hat{v} = v \circ F$

$$\Rightarrow \int_E v \, ds = \frac{|E|}{|\hat{E}|} \int_{\hat{E}} \hat{v} \, d\hat{s}$$

$\mu_{\hat{E}}^k : L^2(\hat{E}) \to P_k(\hat{E})$

$$\int_{\hat{E}} (\hat{v} - \mu_{\hat{E}}^k \hat{v}) \hat{q} \, d\hat{s} = 0 \quad \forall \hat{q} \in P_k(\hat{E}), v \in L^2(\hat{E})$$

For any $v \in L^2(E), q \in P_k(E)$

$$0 = \int_E (v - \mu_E^k v) q \, ds = \frac{|E|}{|\hat{E}|} \int_{\hat{E}} (\hat{v} - \widehat{\mu_E^k v}) \hat{q} \, ds$$

$\hat{v} \in L^2(\hat{E}), \hat{q} \in P_k(\hat{E}) \Rightarrow \widehat{\mu_E^k v} = \mu_{\hat{E}}^k \hat{v}$

$$\int_E v(z-\mu_E^k z)\,ds = \frac{|E|}{|\hat{E}|}\int_{\hat{E}} \hat{v}\,(\hat{z}-\mu_{\hat{E}}^k \hat{z})\,d\hat{s}$$

$$= \frac{|E|}{|\hat{E}|}\int_{\hat{E}} (\hat{v}-c)(\hat{z}-\mu_{\hat{E}}^k)\,d\hat{s} \quad \forall c \in P_0(\hat{T})$$

$$\leq \frac{|E|}{|\hat{E}|}\|\hat{v}-c\|_{0,\hat{E}}\,\|\hat{z}-\mu_{\hat{E}}^k \hat{z}\|_{0,\hat{E}}$$

- $\mu_{\hat{E}}^k \hat{q} = \hat{q} \quad \forall \hat{q} \in P_k(\hat{E})$
- $\|\mu_{\hat{E}}^k \hat{z}\|_{0,\hat{E}} \leq \|\hat{z}\|_{0,\hat{E}} \quad \forall \hat{z} \in L^2(\hat{E})$

$$\int_{\hat{E}} (\mu_{\hat{E}}^k \hat{z})\,\hat{q}\,ds = \int_{\hat{E}} \hat{z}\,\hat{q}\,d\hat{s} \quad \forall \hat{q} \in P_k(\hat{E})$$

Let $\hat{q} := \mu_{\hat{E}}^k \hat{v} \implies \|\mu_{\hat{E}}^k \hat{z}\|_{0,\hat{E}}^2 \leq \|\hat{z}\|_{0,\hat{E}}\|\mu_{\hat{E}}^k \hat{v}\|_{0,\hat{E}}$

- $\|\hat{z}-\mu_{\hat{E}}^k \hat{z}\|_{0,\hat{E}} = \|\hat{z}+\hat{q}-\mu_{\hat{E}}^k(\hat{z}+\hat{q})\|_{0,\hat{E}} \quad \forall \hat{q} \in P_k(\hat{T})$

$$\leq \|\hat{z}+\hat{q}\|_{0,\hat{E}} + \|\mu_{\hat{E}}^k(\hat{z}+\hat{q})\|_{0,\hat{E}}$$

$$\leq 2\|\hat{z}+\hat{q}\|_{0,\hat{E}}$$

$$\leq C\|\hat{z}+\hat{q}\|_{1,\hat{T}}$$

(Remark from Trace Theorem)

- $\|\hat{v}-c\|_{0,\hat{E}} \leq C\|\hat{v}-c\|_{1,\hat{T}}$

$$\implies \left|\int_E v(z-\mu_E^k z)\,ds\right| \leq C h_E \|\hat{v}-c\|_{1,\hat{T}}\,\|\hat{z}+\hat{q}\|_{1,\hat{T}} \quad \forall c \in P_0(\hat{T}),\ \forall \hat{q} \in P_k(\hat{E})$$

$$\implies \text{———''———} \leq C h_E \inf_{c \in P_0(\hat{T})} \|\hat{v}-c\|_{1,\hat{T}} \inf_{\hat{q} \in P_k(\hat{E})} \|\hat{z}+\hat{q}\|_{1,\hat{T}}$$

$$\leq \bar{C} h_E |\hat{v}|_{1,\hat{T}} |\hat{z}|_{k+1,\hat{T}}$$

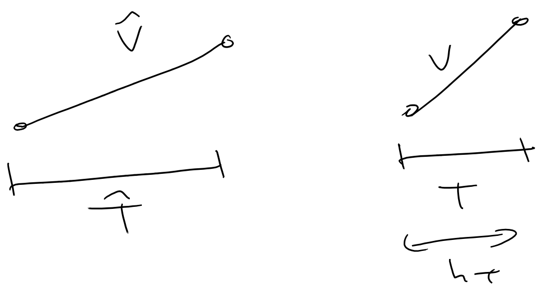
$$\leq \bar{C} h_E^{k+1} |v|_{1,T} |z|_{k+1,T}$$

$|\hat{z}|_{k+1,\hat{T}} \le C h_T^k |z|_{k+1,T}$

"Scaling argument":

$|\hat{v}|_{1,\hat{T}} \le C\, |v|_{1,T}$

$$\Gamma \quad |z|_{k+1,T} \text{ "=" } \left[\int_T \left(\frac{\partial^{k+1} z}{\partial x^{k+1}}\right)^2\right]^{1/2} \sim (\underbrace{h_T^2}_{\sim |T|})^{1/2} \left(\frac{1}{h_T}\right)^{k+1} = h_T^{-k}$$



$$v' \sim \frac{1}{h_T} \qquad \left(\int_T (v')^2\right)^{1/2} \sim \left[h_T \left(\frac{1}{h_T}\right)^2\right]^{1/2} \sim h_T^{-1/2}$$

Now, return to the error analysis:

$$|u-u_h|_{1,h} \le 2 \inf_{v_h \in V_h} |u-v_h|_{1,h} + \sup_{w_h \in V_h} \frac{\sum_{E \in \mathcal{E}_h} \int_E \frac{\partial u}{\partial n_E} [\![w_h]\!]_E \, ds}{|w_h|_{1,h}}$$

$$\int_E \frac{\partial u}{\partial n_E} [\![w_h]\!] \, ds = \int_E \left(\frac{\partial u}{\partial n_E} - \mu_E^{k-1} \frac{\partial u}{\partial n_E}\right) [\![w_h]\!] \, ds$$

$$= \int_E (\ \cdots\ ) w_h|_T \, ds - \int_E (\cdots) w_h|_{\tilde{T}} \, ds$$

$$\le C h_E^k |w_h|_{1,T} \left|\frac{\partial u}{\partial n_E}\right|_{k,T} + C h_E^k |w_h|_{1,\tilde{T}} \left|\frac{\partial u}{\partial n_E}\right|_{k,\tilde{T}}$$

$$\le \tilde{C} h_E^k |u|_{k+1,T \cup \tilde{T}} \left(|w_h|_{1,T} + |w_h|_{1,\tilde{T}}\right)$$

$$\left(\frac{\partial u}{\partial n} = (\nabla u) \cdot n_E\right)$$

$$\Rightarrow \sum_{E \in \mathcal{E}_h} \int_E \frac{\partial u}{\partial n_E} [\![w_h]\!]_E \, ds \le C h^k \left(\sum_{E \in \mathcal{E}_h} |u|^2_{k+1, T_E \cup \tilde{T}_E}\right)^{1/2}$$

$$\times \left(\sum_{E \in \mathcal{E}_h} \left(|w_h|^2_{1,T_E} + |w_h|^2_{1,\tilde{T}_E}\right)\right)^{1/2}$$

$$\le \tilde{C} h^k |u|_{k+1,\Omega} |w_h|_{1,h}$$

Since $V_h \supset \tilde{V}_h := \{ v_h \in C(\bar{\Omega}) \cap H_0^1(\Omega) ;\ v|_T \in P_k(T)\ \forall T \in T_h \}$

$$\inf_{v_h \in V_h} |u - v_h|_{1,h} \leq \inf_{v_h \in \tilde{V}_h} |\underbrace{u - v_h}_{\in H^1(\Omega)}|_{1,\Omega} \leq |u - \underset{\substack{\uparrow \\ \text{Lagrange int.}}}{\Pi_h} u|_{1,\Omega} \leq C h^k |u|_{k+1,\Omega}$$

$$\Rightarrow |u - u_h|_{1,h} \leq C h^k |u|_{k+1,\Omega} \quad (\text{if } u \in H^{k+1}(\Omega)).$$