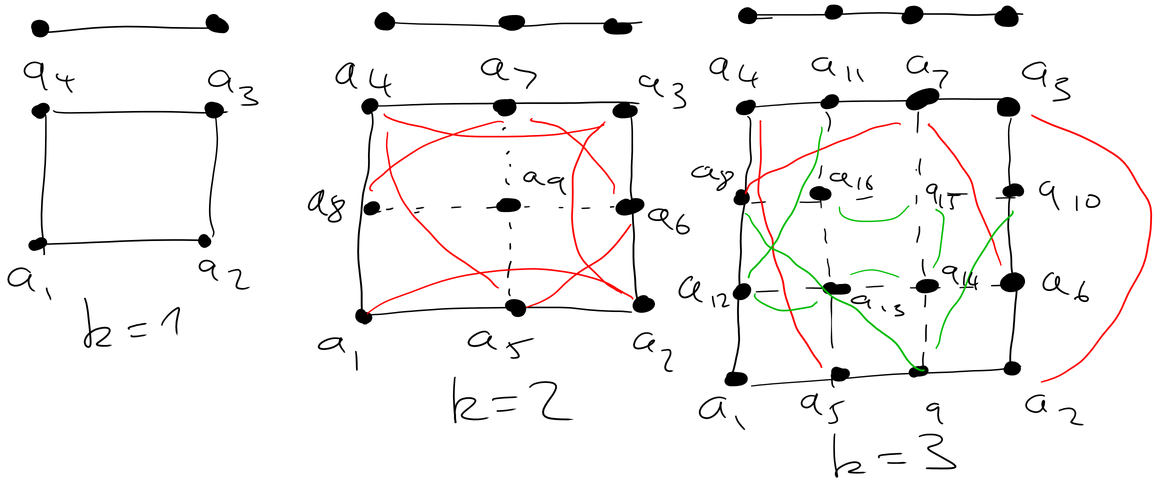
Finite elements on n-rectangles

Define $Q_k = \left\{ \sum_{\substack{\alpha \\ \alpha_i \le k, i=1..n}} \gamma_\alpha x^\alpha : \gamma_\alpha \in \mathbb{R} \right\}$

Ex: $Q_1 = \text{span}\{1, x, y, xy\}$ in 2D-bilinear form

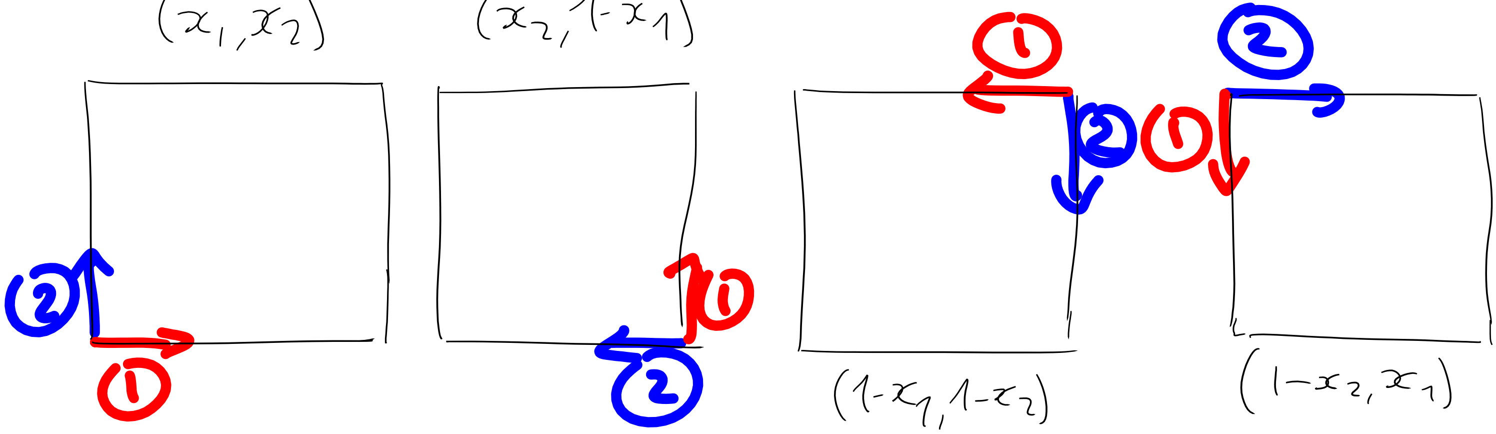


Equidistant points

Tensor product of equidistant points

Consider $n=2$, $a_1, \dots, a_4$ are vertices of the unit square: $a_1=(0,0)$, $a_2=(1,0)$, $a_3=(1,1)$, $a_4=(0,1)$

Coordinates of any x can be written with respect to $a_1, \dots, a_4$ by defining 4 coordinate systems

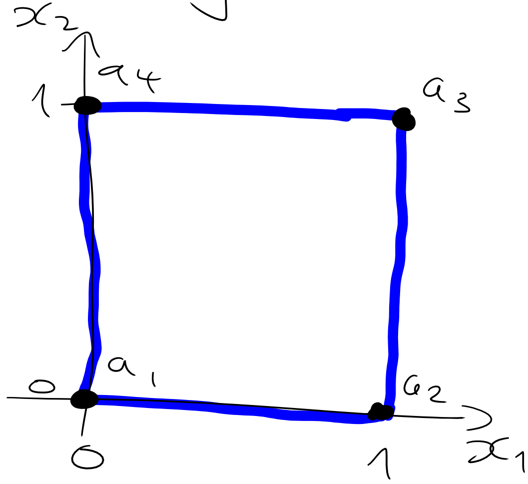


We can introduce notation $x_3 = (1-x_1)$, $x_4 = 1-x_2$

$\Rightarrow (x_1, x_2) \quad (x_2, x_3) \quad (x_3, x_4) \quad (x_4, x_1)$

rotation of coordinate system correspond to circular permutations of indices; therefore, this simplifies definitions of basis functions for rectangular elements.

<u>Ex</u>: Q_1 basis functions



$p_1 = (1-x_1)(1-x_2)$

$p_2 = x_1(1-x_2)$

$p_3 = x_1 x_2$

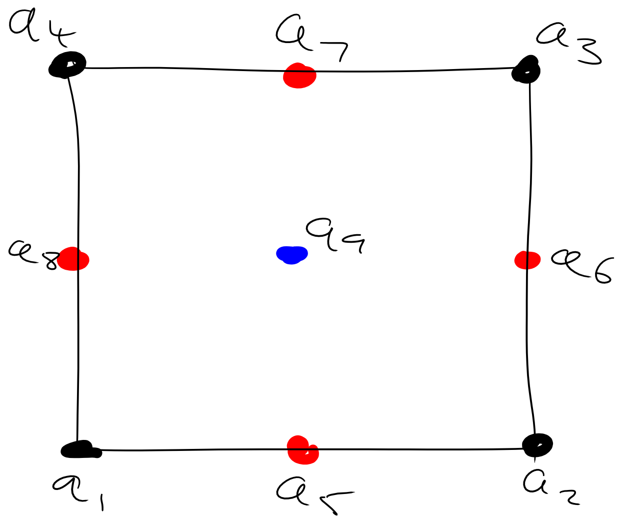
$p_4 = (1-x_1)x_2$

which can be written as

$p_1 = x_3 x_4, \; p_2 = x_4 x_1, \; p_3 = x_1 x_2, \; p_4 = x_2 x_3$

So sufficient to define formula for <u>one</u> basis function and the remaining are defined by circular permutation of indices.

<u>Ex</u>: Q_2 only need to define 3 basis functions

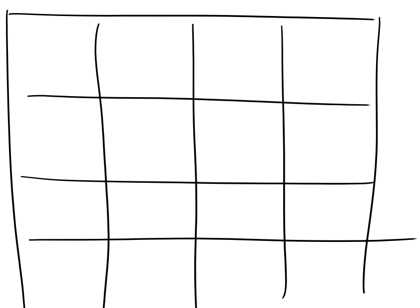


$p_1 = x_3(2x_3-1)x_4(2x_4-1)$

$p_5 = -4x_3(x_3-1)x_4(2x_4-1)$

$p_9 = 16 x_1 x_2 x_3 x_4$

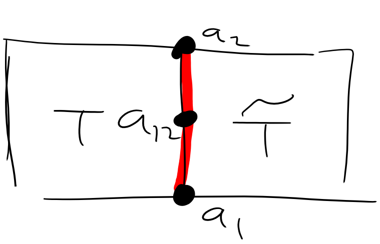
X_h defined analogously as for simplices.



$X_h \subset C(\bar{\Omega})$:

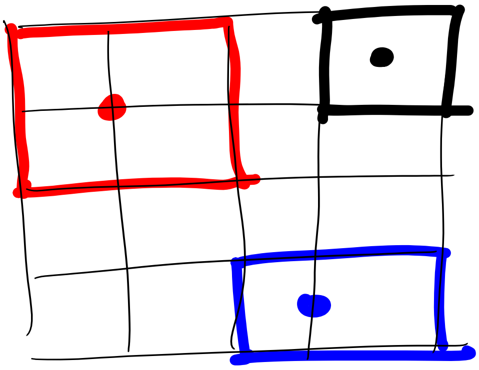
Consider any interior edge E of the triangulation and let $T, \tilde{T}$ be adjacent elements. Let $v \in X_h$ and

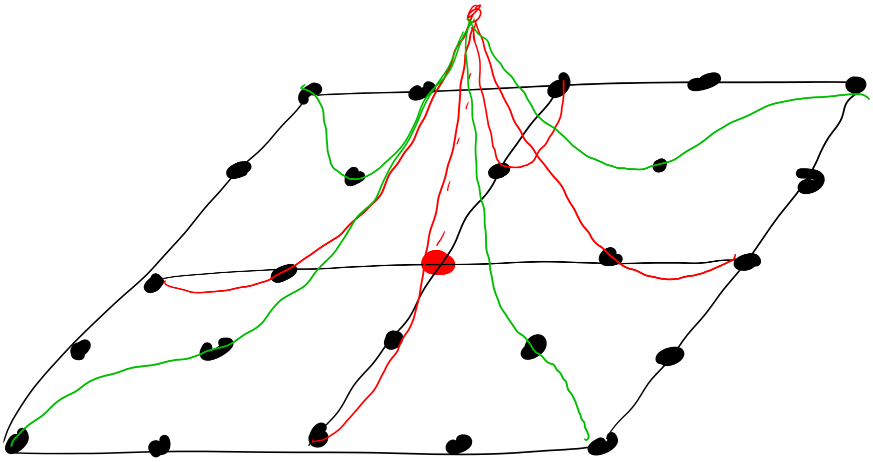
$$p = (v|_T)|_E - (v|_{\tilde{T}})|_E \in P_2(E)$$

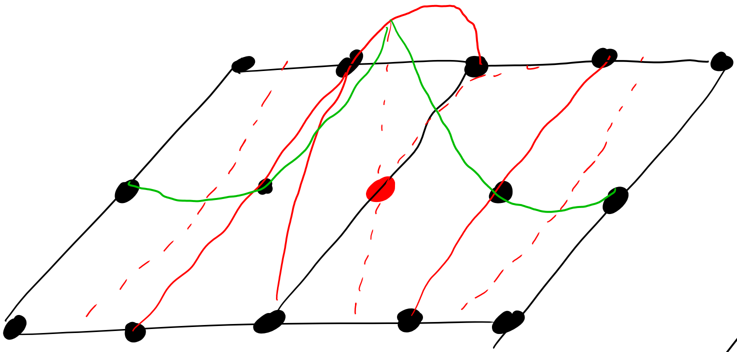


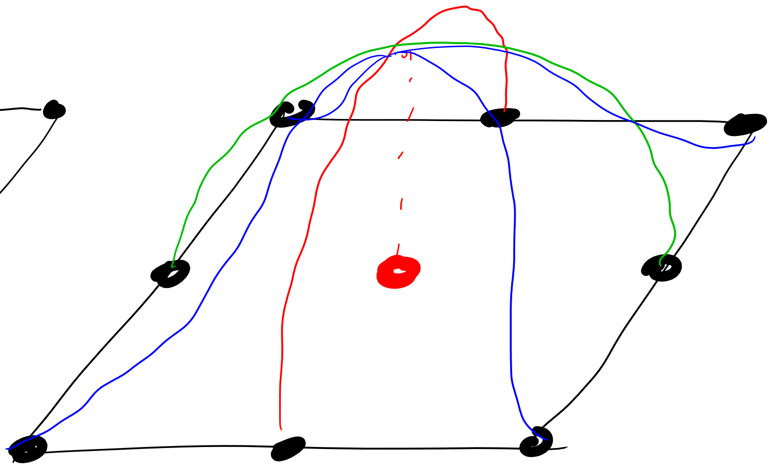
$p(a_1) = p(a_2) = p(a_{12}) = 0$

$\Rightarrow p \equiv 0 \Rightarrow v$ continuous









Again have reduced basis by removing interior nodes of lattice / interior basis. ($k \geq 2$)

Here $P_2 \subset Q_2'(T)$, $P_3 \subset Q_3'(T)$

where Q_k' is reduced basis on n-rectangles.

Hermite Finite Elements

Theorem Let T be an n-simplex with vertices $a_1, \dots, a_{n+1}$ and let $a_{ijk} = \frac{1}{3}(a_i + a_j + a_k)$, $1 \le i < j < k \le n+1$. Then, any polynomial from the space P_3 is uniquely determined by its values at the vertices $a_1, \dots, a_n$, its values at the points a_{ijk}, $1 \le i < j < k \le n+1$, and the values of its $\underline{n}$ first derivatives at the vertices $a_1, \dots, a_{n+1}$.

Proof Show that for any $p \in P_3$,

$$p = \sum_{i=1}^{n+1} \underbrace{\boxed{\Big(-2\lambda_i^3 + 3\lambda_i^2 - 7\lambda_i \sum_{\substack{j<k \\ j,k \neq i}} \lambda_j \lambda_k\Big)}}_{P_i} \underbrace{\boxed{p(a_i)}}_{\text{values at vertices}}$$

$$+ \sum_{i<j<k} \underbrace{\boxed{27 \lambda_i \lambda_j \lambda_k}}_{P_{ijk}} \underbrace{\boxed{p(a_{ijk})}}_{\text{values at interior/face points}}$$

$$+ \sum_{i \neq j} \underbrace{\boxed{\lambda_i \lambda_j (2\lambda_i + \lambda_j - 1)}}_{P_{ij}} \underbrace{\boxed{\nabla p(a_i) \cdot (a_j - a_i)}}_{\text{derivatives at vertices}}$$

Basis function

Consider $\alpha_i, \alpha_{ijk}, \alpha_{ij} \in \mathbb{R}$ and let

$$\tilde{p} = \sum_{i=1}^{n+1} \alpha_i P_i + \sum_{i<j<k} \alpha_{ijk} P_{ijk} + \sum_{i \neq j} \alpha_{ij} P_{ij}$$

Show α_i, α_{ijk} & α_{ij} can be shown as values required for Hermite

Then, $\tilde{p} \in P_3$ and

$$\tilde{p}(a_\ell) = \sum_{i=1}^{n+1} \alpha_i \underbrace{P_i(a_\ell)}_{=\delta_{\ell j}} = \alpha_\ell \qquad \tilde{p}(a_{\ell m s}) = \alpha_{\ell m s}$$

$$\tilde\nabla(a_\ell) = \sum_{i=1}^{n+1} \alpha_i \Big[6\underbrace{(-\lambda_i^2+\lambda_i)}_{=0}\nabla\lambda_i - 7\nabla \sum_{\substack{j<k\\ j,k\neq i}} \underbrace{\lambda_i\lambda_j\lambda_k}_{=0} \Big]$$

$$+\sum_{i\neq j}\Big[(\lambda_i\nabla\lambda_j+\lambda_j\nabla\lambda_i)(2\lambda_i+\lambda_j-1)+\underbrace{\lambda_i\lambda_j}_{=0}(2\nabla\lambda_i+\nabla\lambda_j)\Big]\alpha_{ij}$$

$$+\nabla\sum_{i<j<k} 27\underbrace{\lambda_i\lambda_j\lambda_k}_{=0}\alpha_{ijk}$$

$$= \sum_{i\neq j}\big[\delta_{i\ell}\nabla\lambda_j+\delta_{j\ell}\nabla\lambda_i\big](2\delta_{i\ell}+\delta_{j\ell}-1)\big]\alpha_{ij}$$

$$= \sum_{\substack{j=1\\ j\neq \ell}}^{n+1} \nabla\lambda_j\,\alpha_{\ell j}$$

$$\tilde\nabla(a_\ell)\cdot(a_k-a_\ell) = \sum_{\substack{j=1\\ j\neq\ell}}^{n+1} \underbrace{\nabla\lambda_j\cdot(a_k-a_\ell)}_{\lambda_j(a_k)-\underbrace{\lambda_j(a_\ell)}_{=0}=\delta_{jk}}\alpha_{\ell k} = \alpha_{\ell k}$$

$(k\neq\ell)$

$\Rightarrow$ If $\tilde p = 0$, then all coefficient $\alpha_i, \alpha_{ijk}, \alpha_{ij}$ vanish

$\Rightarrow$ linearly independent $\{p_i\}, \{p_{ij}\}, \{p_{ijk}\}$

Number of functions p_0, p_{ij}, p_{ijk} is

$$(n+1)+\binom{n+1}{3}+(n+1)n = \frac{n+1}{6}\big(6+n(n-1)+6\big)$$

$$=\frac{n+1}{6}(n^2+5n+6)$$

$$=\binom{n+3}{3} = \dim P_3$$

$\Rightarrow \{p_0\}, \{p_{ij}\}, \{p_{ijk}\}$ are basis of P_3 $\square$

- Also have reduced form
- Can define gradient in direction of edge or coordinates

- FE space for Hermite n-simplex
  X_h consists of piecewise P_3 functionals which are continuous at the vertices, at the points a_{ijk} (if $n>2$) and continuous first derivatives at vertices.
- Degrees of freedom:
  $$\Sigma_h = \left\{ v_h(z), z \in N_h^V \cup N_h^C, \frac{\partial v_h}{\partial x_1}(z), \dots \frac{\partial v_h}{\partial x_n}(z), z \in N_h^V \right\}$$
  where N_h^V set of vertices and N_h^C set of barycentres (elements in 2D, faces in 3D)
- Basis of X_h (2D): $N_h^V = \{z_i\}_{i=1}^J$, $N_h^C = \{z_i\}_{i=J+1}^L$

  basis functions are functions
  $$\varphi_i \in X_h, i=1,\dots,L \quad \& \quad \varphi_i^1, \varphi_i^2 \in X_h, i=1,\dots,J.$$
  satisfying
  - $\varphi_i(z_j) = \delta_{ij} \quad i,j=1,\dots,L$ $\qquad$ $\varphi_i(z_i)=1$ all others vanish for φ_i

    $\frac{\partial \varphi_i}{\partial x_1}(z_j) = \frac{\partial \varphi_i}{\partial x_2}(z_j) = 0 \quad i=1,\dots,L, j=1,\dots,J$
  - $\varphi_i^1(z_j) = 0 \quad i=1,\dots,J, j=1,\dots,L$ $\qquad$ $\frac{\partial \varphi_i^1}{\partial x_1}(z_j)=1$ all others vanish

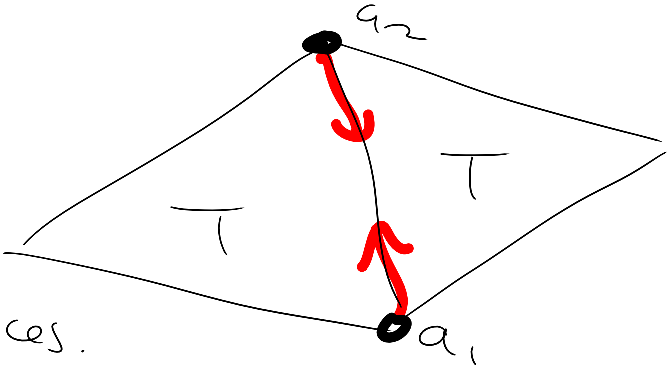
    $\frac{\partial \varphi_i^1}{\partial x_1}(z_j) = \delta_{ij} \quad \frac{\partial \varphi_i^1}{\partial x_2}(z_j) = 0 \quad i,j=1,\dots,J$
  - $\varphi_i^2(z_j) = 0 \quad i=1,\dots,J, j=1,\dots,L$

    $\frac{\partial \varphi_i^2}{\partial x_1}(z_j) = 0 \quad \frac{\partial \varphi_i^2}{\partial x_2}(z_j) = \delta_{ij} \quad i,j=1,\dots,J$

$\frac{\partial \varphi_i^1}{\partial x_2}(z_i) = 1$, all others vanish

Show X_h are continuous in 2D.

Any $v_h \in X_h$ and interior edge E of $\mathcal{T}_h$. Let $T, \tilde{T}$ be adjacent triangles and show v_h continuous across edge E



- Restriction $v_h|_T$ determined on T by vertex values, barycentre value, & first derivatives at vertices.

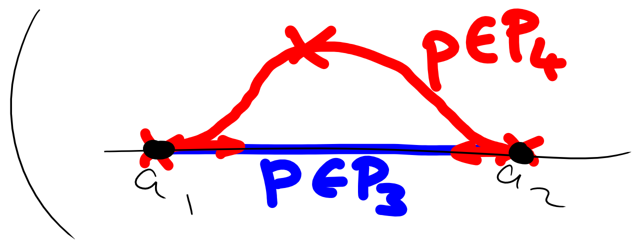
$\Rightarrow$ on E have first derivative of $v_h|_T$ in direction of E at a_1 & a_2 + values at a_1 & a_2

(same for $\tilde{T}$)

Then, define $p := (v_h|_T)|_E - (v_h|_{\tilde{T}})|_E$; then $p \in P_3(E)$

and $p(a_1) = p(a_2) = p'(a_1) = p'(a_2) = 0$.

$\Rightarrow p \equiv 0$



$\Rightarrow p$ continuous on E

$\Rightarrow v_h \in C(\bar{\Omega}) \Rightarrow v_h \in H^1(\Omega)$.