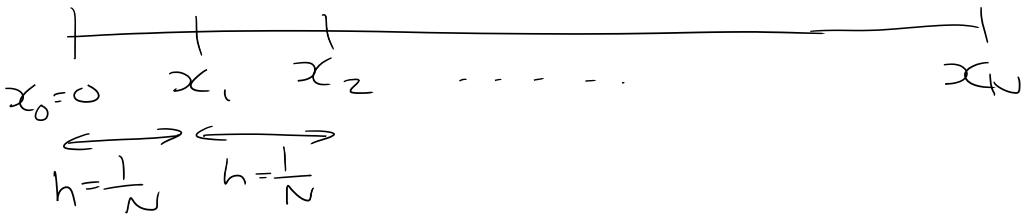
Comparison of FDM & FEM

Consider the one-dimensional proble

$$-\varepsilon u'' + b u' = f \quad \text{on } (0,1)$$
$$u(0) = u(1) = 0$$

where ε, b, f are given constants, $\varepsilon > 0$.

Subdivide the interval $(0,1)$ into N uniform subintervals of length $h = \frac{1}{N}$.



$x_i = ih$
$i = 0, \dots, N$

Discretise the problem using

a) finite difference method using central differences

b) finite element method with space of continuous linear piecewise functions (P_1)

FDM

$$u_i \approx u(x_i) \qquad u_0 = u_N = 0$$

$$u(x_{i\pm1}) = u(x_i) \pm u'(x_i)h + \tfrac{1}{2}u''(x_i)h^2 - \tfrac{1}{6}u'''(x_i)h^3 + \tfrac{1}{24}u^{(4)}(x_i)h^4 \dots$$

$$\Rightarrow u(x_{i+1}) - u(x_{i-1}) = 2u'(x_i)h + O(h^3)$$

$$u(x_{i+1}) + u(x_{i-1}) = 2u(x_i) + u''(x_i)h^2 + O(h^4)$$

$$\Rightarrow u'(x_i) \approx \frac{u_{i+1} - u_{i-1}}{2h}, \quad u''(x_i) \approx \frac{u_{i+1} - 2u_i + u_{i-1}}{h^2}$$

$$\Rightarrow -\varepsilon \frac{u_{i+1} - 2u_i - u_{i-1}}{h^2} + b\frac{u_{i+1} - u_{i-1}}{2h} = f \qquad i = 1, \dots, N-1$$

$$\Rightarrow \left(-\frac{\varepsilon}{h^2} - \frac{b}{2h}\right)u_{i-1} + \frac{2\varepsilon}{h^2}u_i + \left(-\frac{\varepsilon}{h^2} + \frac{b}{2h}\right)u_{i+1} = f \qquad i = 1, \dots, N-1$$

$$\Rightarrow \begin{bmatrix} \frac{2\varepsilon}{h^2} & -\frac{\varepsilon}{h^2}+\frac{b}{2h} & & 0 \\ -\frac{\varepsilon}{h^2}-\frac{b}{2h} & \frac{2\varepsilon}{h^2} & \ddots & \\ & \ddots & \ddots & -\frac{\varepsilon}{h^2}+\frac{b}{2h} \\ 0 & & -\frac{\varepsilon}{h^2}+\frac{b}{2h} & \frac{2\varepsilon}{h^2} \end{bmatrix} \begin{bmatrix} u_1 \\ \vdots \\ u_{N-1} \end{bmatrix} = f, \quad u_0 = u_N = 0$$

FEM

Weak formulation: Find $u \in H_0^1(0,1)$ such that

$$a(u,v) = (f,v) \qquad \forall v \in H_0^1(0,1)$$

where, $f = \text{constant}$,

$$a(u,v) = \varepsilon(u',v') + b(u',v)$$

$$(f,v) = \int_0^1 f v \, dx$$

↳ exists unique weak solution $(\Omega = (0,1))$

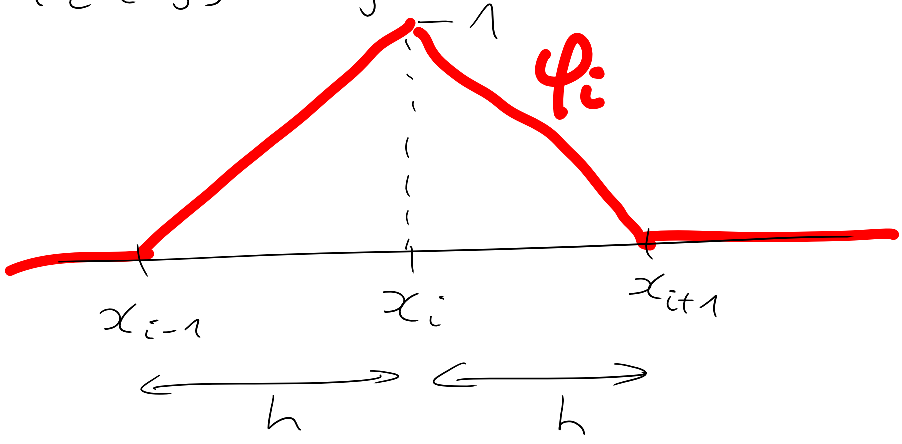
Define finite dimensional subspace of $H_0^1(\Omega)$:

$$H^1(\Omega) \supset X_h = \{ v \in C([0,1]),\ v|_{[x_i, x_{i+1}]} \in P_1(x_i, x_{i+1}),\ i = 0, \dots, N-1 \}$$

Define $V_h = X_h \cap H_0^1(\Omega) := \{ v \in X_h,\ v(0) = v(1) = 0 \}$

$\dim V_h = N-1$, the basis of the space V_h is the set of piecewise continuous linear functionals $\{\varphi_i\}_{i=1}^{N-1} \subset V_h$ such that

$$\varphi_i(x_j) = \delta_{ij}, \quad i,j = 1, \dots, N-1 \quad \text{(hat functions)}$$



$$V_h = \text{span}\{\varphi_i\}_{i=1}^{N-1}$$

Define $u_h = \sum_{i=1}^{N-1} u_j \varphi_j = \sum_{i=1}^{N-1} u_h(x_j)\varphi_j$ as $u_h(x_j) = v_j$

Construct FE formulation: Find $u_h \in V_h$ such that

$$a(u_h, v_h) = (f, v_h) \quad \forall v_h \in V_h.$$

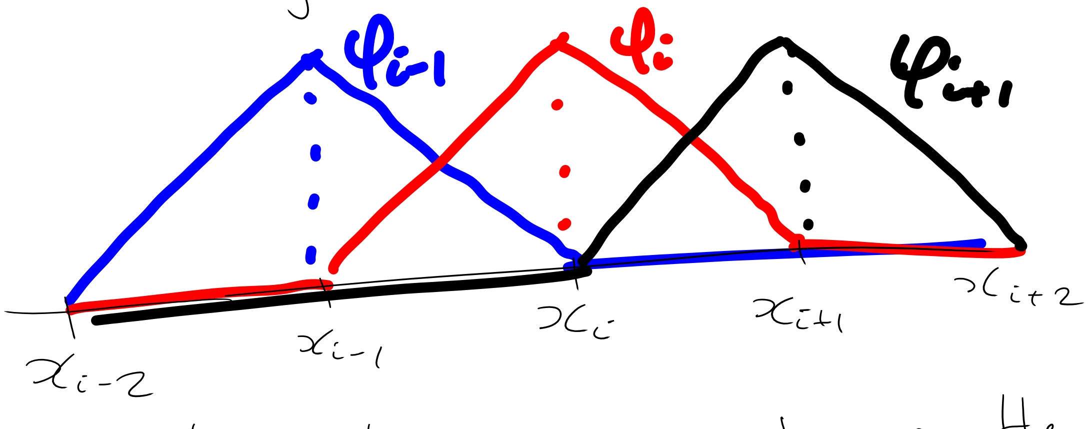
Discretised problem equivalent to solution of the linear system

$$\sum_{j=1}^{N} a(\varphi_j, \varphi_i) u_j = (f, \varphi_j) \qquad i = 1, \dots, N-1.$$

From properties of linear form a & RHS (f, v) the discrete problem has a unique solution (matrix positive definite etc). Define elements of matrix $a_{ij} = a(\varphi_j, \varphi_i)$

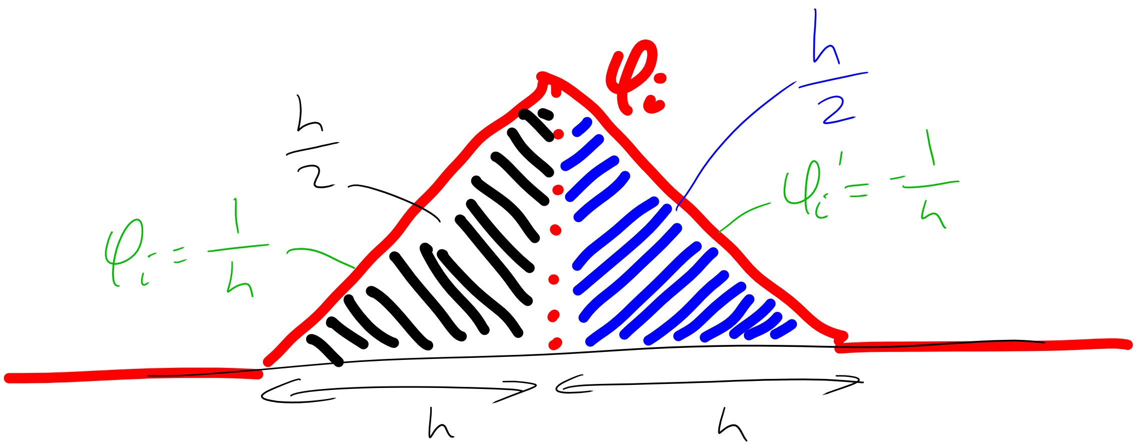
As $\mu(\operatorname{supp} \varphi_i \cap \operatorname{supp} \varphi_j) \neq 0 \iff |i-j| \leq 1$

Then $a_{ij} = 0 \ \forall i,j$ which fulfils $|i-j| > 1$



(FEM supp of basis only in neighbours).

Therefore, only non-zero entries in the ith row are the elements $a_{i,j-1}, a_{ij}, a_{i,i+1}$



Then

$$a_{i,i-1} = a(\varphi_{i-1},\varphi_i) = \int_{x_{i-1}}^{x_i} \varepsilon \varphi_{i-1}' \varphi_i' + b\varphi_{i-1}' \varphi_i \, dx$$

$$= \varepsilon\left(\frac{1}{h}\right)\left(\frac{1}{h}\right)\int_{x_{i-1}}^{x_i} dx + b\left(-\frac{1}{h}\right)\int_{x_{i-1}}^{x_i} \varphi_i dx$$

$$= -\frac{\varepsilon}{h} - \frac{b}{2}$$

$$a_{i,i+1} = a(\varphi_{i+1},\varphi_i) = \int_{x_i}^{x_{i+1}} \varepsilon \varphi_{i+1}' \varphi_i' + b\varphi_{i+1}' \varphi_i dx = -\frac{\varepsilon}{h} + \frac{b}{2}$$

$$a_{i,i} = a(\varphi_i,\varphi_i) = \int_{x_{i-1}}^{x_{i+1}} \varepsilon(\varphi_i')^2 + b\varphi_i'\varphi_i dx$$

$$= \int_{x_{i-1}}^{x_i} \left(-\frac{1}{h}\right)^2 \varepsilon + b\left(-\frac{1}{h}\right)\varphi_i dx + \int_{x_i}^{x_{i+1}} \left(\frac{1}{h}\right)^2 \varepsilon + b\left(\frac{1}{h}\right)\varphi_i dx$$

$$= \frac{2\varepsilon}{h}$$

$$(f,\varphi_i) = f\int_{x_{i-1}}^{x_i} \varphi_i dx = fh$$

Resulting system:

$$\left(-\frac{\varepsilon}{h} - \frac{b}{2}\right)u_{i-1} + 2\frac{\varepsilon}{h}u_i + \left(-\frac{\varepsilon}{h} + \frac{b}{2}\right)u_{i+1} = hf, \quad i = 1, \dots, N-1$$

Resulting linear system equivalent to FDM linear system. Therefore, P_1 1D FEM $\equiv$ FDM with central difference; hence, FEM suffers the same defiencies in the convection-dominated case

↳ oscillations

↳ solution upwinding on first derivatives