Error Estimates in L^2-norm

AVP: $u \in V$ s.t. $a(u,v) = \langle f, v\rangle \quad \forall v \in V$

FEM: $u_h \in V_h \subset V$ s.t. $a(u_h, v_h) = \langle f, v_h\rangle \quad \forall v_h \in V_h$

where $f \in V'$ and V is a Hilbert space with norm $\|\cdot\|_V$. Let H be another Hilbert space with norm $\|\cdot\|_H$ and inner product $(\cdot,\cdot)_H$. Assume

(15) $V \hookrightarrow H$, i.e., $V \subset H$ and $\|v\|_H \le C\|v\|_V \ \forall v \in V$.

Aim is to obtain estimate $\|u-u_h\|_H$ which enables to prove a higher convergence order than for $\|u-u_h\|_V$. In case when $H_0^1(\Omega) \subset V \subset H^1(\Omega)$ typical choice is $H = L^2(\Omega)$.

We introduce an adjoint (or dual) problem to AVP:

(16) Find $\varphi \in V$ such that $a(v,\varphi) = \langle f, v\rangle \ \forall v \in V$.

Lax-Milgram $\Rightarrow \exists!$ solution of (16). In particular, can use $\langle f, v\rangle = (g,v)_H$, where $g \in H$. Since,

$$|(g,v)_H| \le \|g\|_H \|v\|_H \overset{(15)}{\le} C\|g\|_H \|v\|_V \quad \forall v \in V$$

the form $(g,v)_H$ defines an element of V'. Thus,

$$\forall g \in H \quad \exists! \varphi_g \in V \text{ s.t. } a(v, \varphi_g) = (g,v)_H \quad \forall v \in V.$$

Theorem 9 (Aubin-Nitsche Lemma)

Let $V \hookrightarrow H$. Then,

$$\|u-u_h\|_H \le M\|u-u_h\|_V \sup_{g \in H}\left\{\frac{1}{\|g\|_H} \inf_{v_h \in V_h} \|\varphi_g - v_h\|_V\right\}$$

Proof Since $u-u_h \in V$ and $a(u-u_h, v_h)=0$ $\forall v_h \in V_h$ we have that

$$(g, u-u_h)_H = a(u-u_h, \varphi_g) = a(u-u_h, \varphi_g - v_h) \quad \forall v_h \in V_h$$

$$\Rightarrow (g, u-u_h)_H = \inf_{v_h \in V_h} a(u-u_h, \varphi_g - v_h)$$

$$\leq M \|u-u_h\|_V \inf_{v_h \in V_h} \|\varphi_g - v_h\|_V$$

Since $\|u-u_h\|_H = \sup_{g \in H} \frac{(g, u-u_h)_H}{\|g\|_H}$ completes the proof $\square$.

Assume again that the AVP represents a weak formulation of a 2nd order elliptic PDE in Ω with homogeneous Dirichlet BCs; i.e. $V = H_0^1(\Omega)$.

Under assumptions of Theorem 7:

$$\|u-u_h\|_{1,\Omega} \leq C h^k |u|_{k+1,\Omega} \quad \text{for } u \in H^{k+1}(\Omega).$$

Definition The adjoint problem (16) is regular if

1) $\varphi_g \in H^2(\Omega) \quad \forall g \in L^2(\Omega)$
2) $\exists C > 0$ s.t. $\|\varphi_g\|_{2,\Omega} \leq C \|g\|_{0,\Omega} \quad \forall g \in L^2(\Omega)$

Remark Regularity can be proven if coefficients in the PDE are sufficiently smooth and $\partial\Omega$ is sufficiently smooth or Ω is polygonal and convex.

Theorem 10 Let (H1) and (H2) hold and let

$$\hat{\Sigma}_i \subset [H^2(\hat{T}_i)]', \quad i = 1, \dots, M$$

and let (H3), (H4) hold with $Q(\Omega) \supset H^2(\Omega)$.
Let there exist $k \in \mathbb{N}$ such that

$$P_k(\hat{T}_i) \subset \hat{P}_i \subset H^1(\hat{T}_i), \quad i = 1, \dots, M,$$

and let the solution $u \in V$ of the AVP satisfy $u \in H^{k+1}(\Omega)$. Finally, let the adjoint problem 16 be regular. Then, there exists a constant C independent of h such that

$$\|u - u_h\|_{0,\Omega} \le C h^{k+1} |u|_{k+1,\Omega}$$

Proof By Thm 6

$$\|v - \Pi_h v\|_{1,\Omega} \le C h |v|_{2,\Omega} \quad \forall v \in H^2(\Omega) \cap H_0^1(\Omega)$$

Then, for any $g \in L^2(\Omega)$ we have that

$$\inf_{v_h \in V_h} \|\varphi_g - v_h\|_{1,\Omega} \le \|\varphi_g - \Pi_h \varphi_g\|_{1,\Omega} \le C h |\varphi_g|_{2,\Omega}$$
$$\le \tilde{C} h \|g\|_{0,\Omega}$$

$\Rightarrow$ By Thm 9

$$\|u - u_h\| \le M \tilde{C} h \|u - u_h\|_{1,\Omega} \le \overline{C} h^{k+1} |u|_{k+1,\Omega} \quad \square$$

Nonhomogeneous Dirichlet BCs

Consider the problem

(17) $Lu = f$ in Ω, $u = u_b$ on $\partial\Omega$,

where L is a linear differential operator of 2nd order. Then, the weak solution is an element $H^1(\Omega)$

$\Rightarrow$ necessary condition for existence of weak solution is that $\exists \tilde{u}_b \in H^1(\Omega)$ such that $\tilde{u}_b|_{\partial\Omega} = u_b$.

Then, the weak solution of (17) satisfies

$$u \in H^1(\Omega),\ u - \tilde{u}_b \in H_0^1(\Omega),\ a(u,v) = \langle f, v\rangle \ \forall v \in H_0^1(\Omega)$$

$\Rightarrow$ we look for function $\tilde{u} = u - \tilde{u}_b$ satisfying

(18) $\tilde{u} \in H_0^1(\Omega)$, $a(\tilde{u}, v) = \langle \tilde{f}, v\rangle \quad \forall v \in H_0^1(\Omega)$

where $\langle \tilde{f}, v\rangle := \langle f, v\rangle - a(\tilde{u}_b, v)$.

The problem (18) has form of AVP – we can use it for definition of discrete problem like above; i.e, seek $\tilde{u}_h \in V_h$ satisfying

(19) $a(\tilde{u}_h, v_h) = \langle \tilde{f}, v_h\rangle \quad \forall v_h \in V_h$.

Approximate solution of (17) is then $u_h := \tilde{u}_h + \tilde{u}_b$.

Since $u - u_h = \tilde{u} - \tilde{u}_h$ have same error estimates as for $u - u_h$.

However, difficult to construct $\tilde{u}_b$ or compute $(\tilde{u}_b, v)$.

Therefore, replace $\tilde{u}_b$ by some approximation $\tilde{u}_{bh}$.

Then, we find $\overline{u}_h \in V_h$ such that

(20) $a(\overline{u}_h, v_h) = \langle f, v_h\rangle - a(\tilde{u}_{bh}, v_h) \quad \forall v_h \in V_h$

and get approximate solution $u_h := \overline{u}_h + \tilde{u}_{bh}$.

Then,

$$\begin{aligned} u-u_h &= \tilde{u}+\tilde{u}_b-\bar{u}_h-\tilde{u}_{bh} \\ &= (\tilde{u}-\tilde{u}_h)+(\tilde{u}_h-\bar{u}_h)+(\tilde{u}_b-\tilde{u}_{bh}) \end{aligned}$$

where $\tilde{u}_h$ is solution of (19). Since

$$a(\tilde{u}_h-\bar{u}_h, v_h) = a(\tilde{u}_{bh}-\tilde{u}_b, v_h)$$

we have that

$$\|\tilde{u}_h-\bar{u}_h\| \leq \frac{M}{\alpha}\|\tilde{u}_{bh}-\tilde{u}_b\|_{1,\Omega}$$

Thus,

$$\|u-u_h\|_{1,\Omega} \leq \|\tilde{u}-\tilde{u}_h\|_{1,\Omega}+C\|\tilde{u}_b-\tilde{u}_{bh}\|_{1,\Omega}$$

Since have estimates for $\tilde{u}-\tilde{u}_h$, these estimates also hold for $u-u_h$ if $\tilde{u}_{bh}$ defined appropriately.

Natural choice is $\tilde{u}_{bh} \in X_h$. Then, approximate solution does not depend on how $\tilde{u}_{bh}$ defined for "interior" DoFs (values for $\phi(\tilde{u}_{bh})$ for $\phi \in \Sigma_h \setminus \Sigma_h^{\partial\Omega}$).

E.g. we can set

$$(21) \begin{cases} \phi(\tilde{u}_{bh}) = \phi(\Pi_h \tilde{u}_b) = \phi(\tilde{u}_b) & \forall \phi \in \Sigma_h^{\partial\Omega} \\ \phi(\tilde{u}_{bh}) = 0 & \forall \phi \in \Sigma_h \setminus \Sigma_h^{\partial\Omega} \end{cases}$$

if Π_h is defined for $\tilde{u}_b$. In most cases $\phi(\tilde{u}_b)$ depends only on $\tilde{u}_b|_{\partial\Omega} = u_b$ for $\phi \in \Sigma_h^{\partial\Omega}$ (e.g. for all Lagrange FEs). Then $\tilde{u}_{bh}$ in (21) depends only on u_b but not choice of $\tilde{u}_b$.