Theorem 5 Let $(\hat{T}, \hat{P}, \hat{\Sigma})$ be a finite element and $k, m \in \mathbb{N}_0$ and $p, q \in [1, \infty]$ be such that

(10) $\hat{\Sigma} \subset [W^{k+1,p}(\hat{T})]'$, $W^{k+1,p}(\hat{T}) \hookrightarrow W^{m,q}(\hat{T})$,

$P_k(\hat{T}) \subset \hat{P} \subset W^{m,q}(\hat{T})$

Then, there exists a constant $\hat{C}$ such that for any finite element (T, P_T, Σ_T) which is affine equivalent to $(\hat{T}, \hat{P}, \hat{\Sigma})$

(11) $$|v - \Pi_T v|_{m,q,T} \leq \hat{C} |T|^{\frac{1}{q} - \frac{1}{p}} \frac{h_T^{k+1}}{\rho_T^m} |v|_{k+1,p,T} \quad \forall v \in W^{k+1,p}(T)$$

where $\Pi_T v$ is the P_T-interpolation of v.

Proof

Let $\hat{\Sigma} = \{\hat{\Phi}_i\}_{i=1}^N$ and let $\hat{p}_1, \dots, \hat{p}_N \in \hat{P}$ are the basis functions of $(\hat{T}, \hat{P}, \hat{\Sigma})$. Then, the $\hat{P}$-interpolation of any $\hat{v} \in W^{k+1,p}(\hat{T})$ is defined by $\hat{\Pi} \hat{v} = \sum_{i=1}^N \hat{\Phi}_i(\hat{v}) \hat{p}_i$.

Then, $$\|\hat{\Pi} \hat{v}\|_{m,q,\hat{T}} \leq \sum_{i=1}^N |\hat{\Phi}_i(\hat{v})| \, \|\hat{p}_i\|_{m,q,\hat{T}} \quad \text{by } \hat{\Sigma} \subset \dots$$

$$\leq \underbrace{\left(\sum_{i=1}^N \|\hat{\Phi}_i\|_{[W^{k+1,p}(\Omega)]'} \|\hat{p}_i\|_{m,q,\hat{T}}\right)}_{= \hat{C}} \|\hat{v}\|_{k+1,p,\hat{T}}$$

$$= \hat{C} \|\hat{v}\|_{k+1,p,\hat{T}}$$

$\Rightarrow \hat{\Pi} \in \mathcal{L}(W^{k+1,p}(\hat{T}), W^{m,q}(\hat{T}))$

We know that $\hat{\Pi} \hat{p} = \hat{p} \quad \forall \hat{p} \in \hat{P} \supset P_k(\hat{T})$ and

$\widehat{\Pi_T v} = \hat{\Pi} \hat{v} \quad \forall v \in W^{k+1,p}(\hat{T}) \quad \Rightarrow$ (11) follows by Thm. 4. $\square$

Approximating properties of FE spaces

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with Lipschitz continuous boundary and let $\{\mathcal{T}_h\}$ be a family of triangulations satisfying $(\mathcal{T}_h 1)-(\mathcal{T}_h 4)$. Assign finite element (T, P_T, Σ_T) to each set $T \in \mathcal{T}_h$ and let $\{X_h\}$ be the family of corresponding finite element spaces; i.e,

$$X_h = \{v_h \in L^2(\Omega) : v_h|_T \in P_T \ \forall T \in \mathcal{T}_h, \ \phi_{T,i}(v_h|_T) = \phi_{\tilde{T},i}(v_h|_{\tilde{T}})$$
$$\forall T, \tilde{T} \in \mathcal{T}_{h,i}, \ i = 1 \dots, N_h$$

We assume that the parameters h satisfy

$$h_T \leq h \quad \forall T \in \mathcal{T}_h \qquad \left(\text{usually } h = \max_{T \in \mathcal{T}_h} h_T\right)$$

We make the following hypothesis

(H1) The family $\{\mathcal{T}_h\}$ is <u>regular</u>; i.e, there exists a constant σ such that

$$\frac{h_T}{\rho_T} \leq \sigma \quad \forall T \in \mathcal{T}_h \tag{12}$$

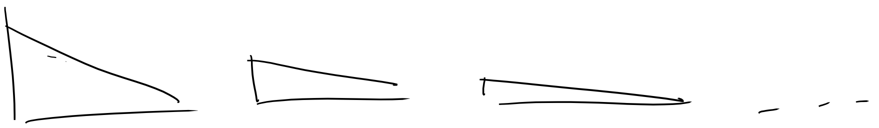
and we have that $h \to 0$

(H2) There exists a finite number of reference finite elements $(\hat{T}_1, \hat{P}_1, \hat{\Sigma}_1) \dots (\hat{T}_M, \hat{P}_M, \hat{\Sigma}_M)$ such that for any $T \in \bigcup \mathcal{T}_h$ the finite element (T, P_T, Σ_T) is affine equivalent to one of the reference elements.

(H3) The space $Q(\Omega) \subset L^1(\Omega)$, on which the X_h-interpolation operator Π_h is defined satisfies

$$(\Pi_h v)|_T = \Pi_T(v|_T) \quad \forall T \in \mathcal{T}_h, \ v \in Q(\Omega).$$

The assumption (12) assumes that elements of the triangulation do not become too "flat" for $h \to 0$. It avoids situations when, e.g., triangles degenerate



anisotropic element

Spaces X_h generally contain discontinuous function or discontinuous derivatives, so we introduce discrete (broken) analogues of the Sobolev norms & seminorms:

$$|v|_{k,p,h} = \Big(\sum_{T \in \mathcal{T}_h} |v|^p_{k,p,T}\Big)^{1/p} \qquad \|v\|_{k,p,h} = \Big(\sum_{T \in \mathcal{T}_h} \|v\|^p_{k,p,T}\Big)^{1/p}$$

$$|v|_{k,\infty,h} = \max_{T \in \mathcal{T}_h} |v|_{k,\infty,T} \qquad \|v\|_{k,\infty,h} = \max_{T \in \mathcal{T}_h} \|v\|_{k,\infty,T}$$

Clearly

$$|v|_{k,p,h} = |v|_{k,p,\Omega} \;\&\; \|v\|_{k,p,h} = \|v\|_{k,p,\Omega} \quad \forall v \in W^{k,p}(\Omega).$$

Theorem 6

Let (H1)–(H3) hold and let $k, l, r \in \mathbb{N}_0$ and $p \in [1,\infty]$ are such that the reference finite elements satisfy

(13) $\hat{\Sigma}_i \subset [W^{k+1,p}(\hat{T}_i)]'$, $P_k(\hat{T}_i) \subset \hat{P}_i \subset W^{r,p}(\hat{T}_i)$, $i = 1, \dots, M$

In addition, let $Q(\Omega) \supset W^{l+1,p}(\Omega)$. Then, there exists a constant C independent of h such that, for any $m, s \in \mathbb{N}_0$ satisfying

$$l \le s \le k \qquad 0 \le m \le \min\{r, s+1\},$$

we have that

(14) $$|v - \Pi_h v|_{m,p,h} \le C h^{s+1-m} |v|_{s+1,p,\Omega} \quad \forall v \in W^{s+1,p}(\Omega)$$

where $\Pi_h v$ is the X_h-interpolation of $\underline{v}$.

Proof Let k, l, v, p, m, s satisfy the assumptions; then,
$\hat{\Sigma}_i \in [W^{s+1,p}(\hat{T}_i)]$ $W^{s+1,p}(\hat{T}_i) \hookrightarrow W^{m,p}(\hat{T}_i)$,
$P_s(\hat{T}_i) \subset \hat{P}_i \subset W^{m,p}(\hat{T}_i)$ $i = 1, \ldots, M$.

Then, by Theorem 5 & (12), for any $T \in \bigcup \mathcal{T}_h$
$$|v - \Pi_T v|_{m,p,T} \leq C h_T^{s+1-m} |v|_{s+1,p,T} \quad \forall v \in W^{s+1,p}(T)$$
Sum over $T \in \mathcal{T}_h$ completes the proof. $\square$

Remark (14) clearly holds with $\|v - \Pi_h v\|_{m,p,h}$

Convergence of discrete solutions

Let us consider a PDE of 2^{nd} order defined in Ω, and, for simplicity, consider homogeneous Dirichlet boundary conditions on whole boundary. Assume weak formulation has form of AVP with $V = H_0^1(\Omega)$.
Consider family of triangulations $\{\mathcal{T}_h\}$ on Ω and family $\{X_h\}$ of corresponding finite element spaces.
We set
$$V_h = \{v_h \in X_h : \phi(v_h) = 0 \quad \forall \phi \in \Sigma_h^{\partial\Omega}\}$$
and assume that $\{\mathcal{T}_h\}$ and finite elements are such that $V_h \subset H_0^1(\Omega)$. Finally, assume that
(H4) The space $Q(\Omega)$ satisfies $\Pi_h v \in V_h$
$$\forall v \in Q(\Omega) \cap H_0^1(\Omega)$$

We now consider conforming discretisations
$$\text{find } u_h \in V_h \text{ such that } a(u_h, v_h) = \langle f, v_h \rangle \; \forall v_h \in V$$

Theorem 7 Let (H1) – (H4) hold and let there exist $k \in \mathbb{N}_0$ such that

$$\hat{\Sigma}_i \subset [H^{k+1}(\hat{T}_i)]' \quad P_k(\hat{T}_i) \subset \hat{P}_i \subset H^1(\hat{T}_i), \; i=1,..,M$$

Let $Q(\Omega) \supset H^{k+1}(\Omega)$ and let the solution $u \in V$ of the AVP satisfy $u \in H^{k+1}(\Omega)$. Then, there exists a constant C independent of h such that

$$\|u-u_h\|_{1,\Omega} \leq C h^k |u|_{k+1,\Omega}.$$

Proof Due to Theorem 6 $\|v-\Pi_h v\|_{1,\Omega} \leq C h^k |v|_{k+1,\Omega}$ $\forall v \in H^{k+1}(\Omega) \cap H_0^1(\Omega)$. For $v=u$, theorem follows for Céa's Lemma $\square$

Theorem 8 Let (H1), (H2) hold and let $\exists k \in \mathbb{N}$ such that

$$\hat{\Sigma}_i \subset [W^{k+1,\infty}(\hat{T}_i)]', \quad P_k(\hat{T}_i) \subset P_i \subset H^1(\hat{T}_i), \; i=1,...,M$$

Then, $\lim\limits_{h \to 0} \|u-u_h\|_{1,\Omega} = 0$

Proof It follow from Theorem 5 that, for any $T \in \bigcup \mathcal{T}_h$

$$\|v-\Pi_T v\|_{1,T} \leq \hat{C} |T|^{1/2} h_T^k |v|_{k+1,\infty,T} \quad \forall v \in W^{k+1,\infty}(T)$$

In particular, for $v \in C_0^\infty(\Omega)$ we have $(\Pi_h v)|_T = \Pi_T(v|_T)$ $\forall T \in \mathcal{T}_h$ and hence

$$(*) \quad \|v-\Pi_h v\|_{1,\Omega} = \Big[\sum_{T \in \mathcal{T}_h} \|v-\Pi_T(v|_T)\|_{1,T}^2\Big]^{1/2} \leq \hat{C} |\Omega|^{1/2} h^k |v|_{k+1,\infty,\Omega} \quad \forall v \in C_0^\infty(\Omega)$$

According to definition of V_h $\Pi_h v \in V_h$ for any $v \in C_0^\infty(\Omega)$

Let $\varepsilon > 0$. Since $C_0^\infty(\Omega)$ is dense in V $\exists v_\varepsilon \in C_0^\infty(\Omega)$ s.t $\|u-v_\varepsilon\|_{1,\Omega} < \frac{\varepsilon}{2}$. According to $(*)$ $\exists h_\varepsilon$ such that $\|v_\varepsilon - \Pi_h v_\varepsilon\|_{1,\Omega} < \frac{\varepsilon}{2}$ $\forall h < h_\varepsilon$.

Thus,

$$\inf_{v_h \in V_h} \|u - v_h\|_{1,\Omega} \leq \|u - \Pi_h v_\varepsilon\|_{1,\Omega} \leq \|u - v_\varepsilon\|_{1,\Omega} + \|v_\varepsilon - \Pi_h v_\varepsilon\|_{1,\Omega}$$

$$< \varepsilon \qquad \forall h < h_\varepsilon$$

$$\Rightarrow \lim_{h \to 0} \inf_{v_h \in V_h} \|u - v_h\|_{1,\Omega} = 0$$

and the theorem follows from Céa's Lemma $\square$

<u>Remark</u> $W^{k+1,\infty}(\hat{T}_i) \hookrightarrow C^s(\bar{\Omega})$ for $s \in \{0, \dots k\}$

$\Rightarrow$ assumptions satisfied if the DoFs are not defined using higher derivatives than k (usual case).