Error Estimates & Convergence

Return to the AVP: find $u \in V$ such that

$$a(u,v) = \langle f, v \rangle \quad \forall v \in V$$

where V, a, and f satisfy the assumptions of the AVP.
Let $V_h \subset V$ be finite element spaces and we now assume AVP corresponds to a weak formulation of a PDE. The parameter h corresponds to the size of elements of $\mathcal{T}_h$. We assume that we construct a family $\{V_h\}$, where $h \longrightarrow 0^+$.

For any space V_h we define the discrete problem: Find $u_h \in V_h$ such that

$$a(u_h, v_h) = \langle f, v_h \rangle \quad \forall v_h \in V_h.$$

Aim to construct V_h such that discrete solutions u_h converges to solution u of AVP; i.e,

$$\lim_{h \to 0^+} \|u - u_h\|_V = 0$$

__Theorem (Céa)__ There exists a constant C independent of V_h such that

$$\|u - u_h\|_V \leq C \inf_{v_h \in V_h} \|u - v_h\|_V$$

$\Rightarrow$ A sufficient condition for convergence is that spaces V_h satisfy

$$\forall u \in V \quad \lim_{h \to 0^+} \inf_{v_h \in V_h} \|u - v_h\|_V = 0$$

$\Rightarrow$ Problem of estimating $\|u-u_h\|_V$ reduced to problem of approximating of V by spaces V_h.

Proof

AVP: Find $u \in V$: $\quad a(u,v) = \langle f, v\rangle \quad \forall v \in V$

$$a(u,v_h) = \langle f, v_h\rangle \quad \forall v_h \in V_h (\subset V)$$

Discrete problem:

Find $u_h \in V_h$: $\quad a(u_h, v_h) = (f,v) \quad \forall v \in V_h$

$$a(u-u_h, v_h) = 0 \quad \forall v_h \in V_h$$

$$\begin{aligned}\alpha \|u-u_h\|_V^2 &\le a(u-u_h, u-u_h) \\ &= a(u-u_h, u-v_h) \qquad \forall v_h \in V_h \\ &\le M \|u-u_h\|_V \|u-v_h\|_V\end{aligned}$$

$$\Rightarrow \|u-u_h\|_V \le \frac{M}{\alpha} \|u-v_h\|_V \quad \forall v_h \in V_h$$

$$\Rightarrow \|u-u_h\|_V \le \frac{M}{\alpha} \inf_{v_h \in V_h} \|u-v_h\|_V \qquad \square$$

Error estimates for 2^{nd} order PDEs

Céa: $\|u-u_h\|_V \le C \inf_{v_h \in V_h} \|u-v_h\|_V$

Assume $u \in Q(\Omega)$. Then

$$\|u-u_h\|_V \le C \|u - \Pi_h u\|_V.$$

If the AVP corresponds to weak formulation of a second order PDE, then $V \subset H^1(\Omega)$ and $\|\cdot\|_V = \|\cdot\|_{1,\Omega}$.

Then,

$$\|u-u_h\|_{1,\Omega} \le C \|u - \Pi_h u\|_{1,\Omega}$$

$$= C \left[\sum_{T \in \mathcal{T}_h} \|u - \Pi_h u\|_{1,T}^2\right]^{1/2} = C \left[\sum_{T \in \mathcal{T}_h} \|u - \Pi_T (u|_T)\|_{1,T}^2\right]^{1/2}$$

$\Rightarrow$ estimation of $\|u-u_h\|_{1,\Omega}$ is reduced to problem of estimating error of local interpolation.

Theorem 1 Let $G \subset \mathbb{R}^n$ be a bounded domain with Lipschitz-continuous boundary. Let $k \geq 0$ be a non-negative integer and $p \in [1,\infty]$. Then, there exists a constant C, depending only on G, k, and p such that

① $$\inf_{q \in P_k(G)} \|v+q\|_{k+1,p,G} \leq C |v|_{k+1,p,G} \quad \forall v \in W^{k+1,p}(G).$$

[seminorm $|\cdot|_{k+1,p,\Omega}$ is equivalent to norm on quotient space $W^{k+1,p}(\Omega)/P_k(\Omega)$]

Proof Let $N = \dim P_k(G)$ and $f_1, \ldots, f_N$ basis of the dual space to $P_k(G)$. According to Hahn-Banach theorem, there exists continuous linear functionals on $W^{k+1,p}(G)$ which are extensions of $f_1 \ldots f_N$ and will be denoted by $f_1, \ldots, f_N$ (as well). Then,

② $\forall p \in P_k(G)$: $f_i(p) = 0, i = 1, \ldots, N \Rightarrow p = 0$

We shall show $\exists C > 0$ such that

③ $$\|v\|_{k+1,p,G} \leq C\left(|v|_{k+1,p,G} + \sum_{i=1}^{N} |f_i(v)|\right) \quad \forall v \in W^{k+1,p}(G)$$

Choose $v \in W^{k+1,p}(G)$; then, there exists $q \in P_k(G)$ such that $f_i(v+q) = 0, i = 1, \ldots, N$; then,

$$\|v+q\|_{k+1,p,G} \leq C|v+q|_{k+1,p,G} = C|v|_{k+1,p,G} \quad \Rightarrow ①$$

Just need to prove ③. Lets assume it does not hold. Then, $\forall C > 0 \; \exists v \in W^{k+1,p}(G)$ such that

$$\|v\|_{k+1,p,G} > C\Big(|v|_{k+1,p,G} + \sum_{i=1}^{N} |f_i(v)|\Big).$$

$\Rightarrow \exists \{v_\ell\}_{\ell=0}^{\infty} \subset W^{k+1,p}(G)$ such that $\forall \ell \in \mathbb{N}$

(4) $\|v_\ell\|_{k+1,p,G} = 1, \quad |v_\ell|_{k+1,p,G} + \sum_{i=1}^{N} |f_i(v_\ell)| < \frac{1}{\ell}$

Since $\{v_\ell\}_{\ell=1}^{\infty}$ is bounded in $W^{k+1,p}(G)$, there is a subsequence (again denoted as $\{v_\ell\}$), such that $\{v_\ell\}$ converges to $W^{k,p}(G)$ (follows from Rellich's theorem for $p \in [1,\infty)$ & Arzelà-Ascoli for $p=\infty$).

Thus $\{v_\ell\}$ is a Cauchy sequence in $\|\cdot\|_{k,p,G}$.

Due to (4) $\lim\limits_{\ell\to\infty} |v_\ell|_{k+1,p,G} = 0 \Rightarrow \{v_\ell\}$ Cauchy seq. in $|\cdot|_{k+1,p,G}$

$\Rightarrow$ also in $\|\cdot\|_{k+1,p,G}$

Since $W^{k+1,p}(G)$ is a Banach space, there is $v \in W^{k+1,p}(G)$ such that $\|v - v_\ell\|_{k+1,p,G} \to 0$.

Due to (4)

(5) $\|v\|_{k+1,p,G} = 1, \quad |v|_{k+1,p,G} = 0, \quad \sum_{i=1}^{N} |f_i(v)| = 0$

Then, $|v|_{k+1,p,G} = 0$ implies that $D^\alpha v = 0 \ \forall \alpha, |\alpha| = k+1$

$\Rightarrow v \in P_k(G) \Rightarrow f_i(v) = 0, \ i = 1, \dots, N \Rightarrow v = 0$

$\Rightarrow$ contradiction with (5). $\quad \square$

Theorem 2 Let G and $\hat{G}$ be two bounded domains in $\mathbb{R}^n$ with Lipschitz-continuous boundaries which are affine-equivalent, i.e, $G = F(\hat{G})$, where F is an invertible affine mapping $(F(\hat{x}) = \mathbb{B}\hat{x} + b)$. Let $m \in \mathbb{N}_0$ and $p \in [1, \infty]$. Then, for any $v \in W^{m,p}(G)$, the function $\hat{v} = v \circ F$ belongs to $W^{m,p}(\hat{G})$. Additionally

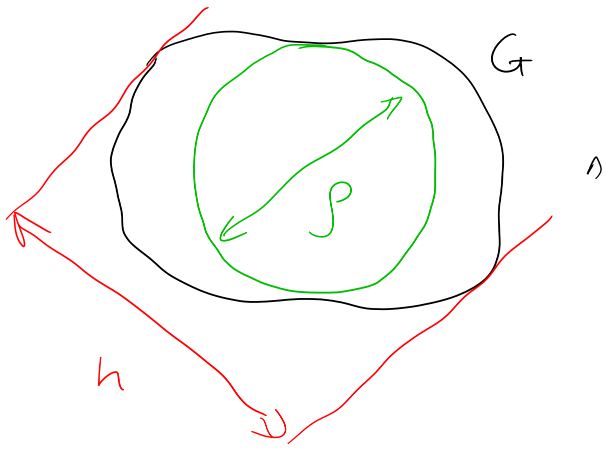
(6) $$|\hat{v}|_{m,p,\hat{G}} \le C \|\mathbb{B}\|^m |\det \mathbb{B}|^{-1/p} |v|_{m,p,G} \quad \forall v \in W^{m,p}(G)$$

(7) $$|v|_{m,p,G} \le C \|\mathbb{B}^{-1}\|^m |\det \mathbb{B}|^{\frac{1}{p}} |\hat{v}|_{m,p,G} \quad \forall \hat{v} \in W^{m,p}(\hat{G})$$

where C depends only on m & n and $\|\mathbb{B}\| = \sup\limits_{x \in \mathbb{R}^n} \frac{|\mathbb{B}x|}{|x|}$.

Proof - See practical

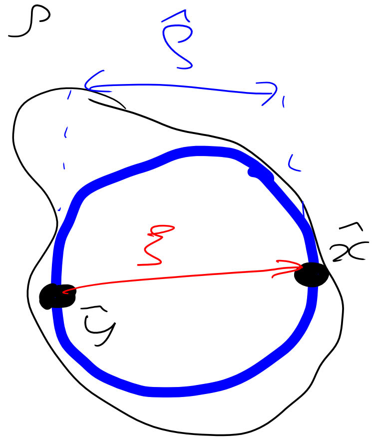
$$(8) \begin{cases} h = \operatorname{diam}(G) \qquad \hat{h} = \operatorname{diam}(\hat{G}) \\ \rho = \sup\{\operatorname{diam}(B) : B \subset G \text{ is a ball}\} \\ \hat{\rho} = \sup\{\operatorname{diam}(\hat{B}) : \hat{B} \subset \hat{G} \text{ is a ball}\} \end{cases}$$



Theorem 3 Let $F(\hat{x}) = \mathbb{B}\hat{x} + b$ be an invertible affine mapping and let $\hat{G}$ and $G = F(\hat{G})$ be two affine-equivalent subsets of $\mathbb{R}^n$. Then,

$$\|\mathbb{B}\| \le \frac{h}{\hat{\rho}}, \qquad \|\mathbb{B}^{-1}\| \le \frac{\hat{h}}{\rho}.$$

Proof $\quad \|B\| = \sup_{\xi\in\mathbb{R}^n} \frac{|B\xi|}{|\xi|} = \sup_{|\xi|=\hat\rho} |B\xi|\frac{1}{\hat\rho}$



If $\xi\in\mathbb{R}^n$ satisfies $|\xi|=\hat\rho$, then there exists points $\hat x,\hat y\in\hat G$ such that $\xi=\hat x-\hat y$.
Then $B\xi = B\hat x - B\hat y = F(\hat x)-F(\hat y)$
and $F(\hat x),F(\hat y)\in G \Rightarrow |B\xi|\le h$

$$\Rightarrow \|B\|\le \frac{h}{\hat\rho}$$

Second inequality follows from $\hat G = F^{-1}(G)$, $F^{-1}(x)=B^{-1}x-B^{-1}b$. □

Theorem 4 Let $\hat G\subset\mathbb{R}^n$ be a bounded domain with Lipschitz continuous boundary and let $k,m\in\mathbb{N}_0$ and $p,q\in[1,\infty]$ be such that $W^{k+1,p}(\hat G)\hookrightarrow W^{m,q}(\hat G)$. Let $\hat\Pi\in\mathcal{L}(W^{k+1,p}(\hat G),W^{m,q}(\hat G))$ satisfying

$$(9)\qquad \hat\Pi p=\hat p \qquad \forall \hat p\in P_k(\hat G)$$

For any domain G which affine-equivalent to $\hat G$ we define Π_G by $\widehat{\Pi_G v}=\hat\Pi \hat v \quad \forall v\in W^{k+1,p}(G)$
(i.e. $\Pi_G v = (\hat\Pi(v\circ F))\circ F^{-1} \quad \forall v\in W^{k+1,p}(G)$)

Then, there exists a constant $\hat C$ depending only on $\hat\Pi$, $\hat G$, and k,m,n,p,q such that

$$|v-\Pi_G v|_{m,q,G} \le \hat C\,|G|^{\frac1q-\frac1p}\frac{h^{k+1}}{\rho^m}|v|_{k+1,p,G} \qquad \forall v\in W^{k+1,p}(G).$$

where h & ρ are defined by 8.

Proof Due to (9) for any $\hat{v} \in W^{k+1,p}(\hat{G})$ and $\hat{p} \in P_k(\hat{G})$

$$\hat{v} - \hat{\Pi}\hat{v} = \hat{v} + \hat{p} - \hat{\Pi}(\hat{v}+\hat{p}) = (I - \hat{\Pi})(\hat{v}+\hat{p})$$

(I - identity - continuous mapping).

$$|\hat{v} - \hat{\Pi} v|_{m,q,\hat{G}} \le \|I - \hat{\Pi}\|_{\mathcal{L}(W^{k+1,p}(\hat{G}), W^{m,q}(\hat{G}))} \times \inf_{\hat{p} \in P_k(\hat{G})} \|\hat{v} + \hat{p}\|_{k+1,p,\hat{G}}$$

$$\le C(\Pi, \hat{G}, k, p, m, q)\, |\hat{u}|_{k+1,p,\hat{G}} \quad \text{(Thm 1)}$$

Since $\hat{v} - \hat{\Pi}\hat{v} = v - \Pi_G v$, from Theorem 2

$$|v - \Pi_G v|_{m,q,G} \le C(m,n)\, \|B^{-1}\|^m |\det B|^{1/q} |\hat{v} - \hat{\Pi}\hat{v}|_{m,q,\hat{G}}$$

$$|\hat{v}|_{k+1,p,\hat{G}} \le C(k,p)\, \|B\|^{k+1} |\det B|^{-\frac{1}{p}} |v|_{k+1,p,G}$$

Combining these results gives

$$|v - \Pi_G v|_{m,q,G} \le C \|B^{-1}\|^m \|B\|^{k+1} |\det B|^{\frac{1}{q} - \frac{1}{p}} |v|_{k+1,p,G}$$

Theorem 3 $\Rightarrow$ $\|B^{-1}\|^m \|B\|^{k+1} \le \frac{\hat{h}^m}{\rho^m} \cdot \frac{h^{k+1}}{\hat{\rho}^{k+1}} = C(\hat{G}) \frac{h^{k+1}}{\rho^{k+1}}$

$$|G| = \int_G dx = \int_{\hat{G} = F^{-1}(G)} \left|\det \frac{DF}{D\hat{x}}\right| d\hat{x} = \int_{\hat{G}} |\det B|\, d\hat{x} = |\hat{G}| \cdot |\det B| \qquad \square$$

Theorem 5 Let $(\hat{T}, \hat{P}, \hat{\Sigma})$ be a finite element and let $k, m \in \mathbb{N}_0$ and $p, q \in [1, \infty]$ be such that

(10) $\hat{\Sigma} \subset [W^{k+1,p}(\hat{T})]'$, $W^{k+1,p}(\hat{T}) \hookrightarrow W^{m,q}(\hat{T})$

$P_k(\hat{T}) \subset \hat{P} \subset W^{m,q}(\hat{T})$

Then, there exists a constant $\hat{C}$ such that for any finite element (T, P_T, Σ_T) which is affine equivalent to $(\hat{T}, \hat{P}, \hat{\Sigma})$

(11) $$|v - \Pi_T v|_{m,q,T} \le \hat{C} |T|^{\frac{1}{q} - \frac{1}{p}} \frac{h_T^{k+1}}{\rho_T^m} |v|_{k+1,p,T} \quad \forall v \in W^{k+1,p}(T)$$

where $\Pi_T v$ is the P_T-interpolation of v.