<u>Definition</u> Two finite elements $(\hat{T}, \hat{P}, \hat{\Sigma})$ and (T, P, Σ) are affine-equivalent if there exists an invertible affine mapping $F: \mathbb{R}^n \to \mathbb{R}^n$ such that

$$T = F(\hat{T})$$
$$P = \{p : T \to \mathbb{R} : p = \hat{p} \circ F^{-1}, \hat{p} \in \hat{P}\}$$
$$\Sigma = \{\phi : \phi(p) = \hat{\phi}(p \circ F), \hat{\phi} \in \hat{\Sigma}\}$$

<u>Example</u> Consider cubic Hermite n-simplex from Fig 8.1. Consider

$$\Sigma = \{p(a_i), i = 1, \dots, n+1 ; p(a_{ijk}), 1 \le i < j < k \le n+1$$
$$\nabla p(a_i) \cdot (a_j - a_i), \; i, j = 1, \dots, n+1, \; i \neq j\}$$

- Show (T, P_T, Σ_T) and $(\hat{T}, P_{\hat{T}}, \Sigma_{\hat{T}})$ affine-equivalent

$F(\hat{x}) = B\hat{x} + b$ such that $T = F(\hat{T})$ & already shown that $P_T = P_{\hat{T}}$

- Number vertices of T $(a_1, \dots, a_{n+1})$ and $\hat{T}(\hat{a}_1, \dots \hat{a}_{n+1})$ such that $F(\hat{a}_i) = a_i, i = 1, \dots, n+1$.

Need to verify that $\Sigma_T = \{\phi : \phi(p) = \hat{\phi}(p \circ F), \phi \in \Sigma_{\hat{T}}\}$

We showed previously that

$$\phi_i(p) = p(a_i) = (p \circ F)(\hat{a}_i) = \hat{\phi}_i(p \circ F)$$
$$\phi_{ijk}(p) = p(a_{ijk}) = (p \circ F)(\hat{a}_{ijk}) = \hat{\phi}_i(p \circ F)$$

Need to consider if $\phi_{ij}(p) = \hat{\phi}_{ij}(p \circ F)$ where

$$\phi_{ij}(p) = \nabla p(a_i) \cdot (a_j - a_i), \quad \hat{\phi}_{ij}(p \circ F) = \hat{\nabla}(p \circ F)(\hat{a}_i) \cdot (\hat{a}_j - \hat{a}_i)$$

$$\hat{\nabla}(p \circ F)(\hat{a}_i)\cdot(\hat{a}_j - \hat{a}_i) = \sum_{k=1}^{n} \frac{\partial}{\partial \hat{x}_k}(p \circ F)(\hat{a}_i)(\hat{a}_j - \hat{a}_i)_k$$

$$= \sum_{k,l=1}^{n} \frac{\partial p}{\partial x_l}(F(\hat{a}_i))\underbrace{\frac{\partial F_l}{\partial \hat{x}_k}(\hat{a}_i)}_{\equiv B_{lk}}(\hat{a}_j - \hat{a}_i)_k$$

$$= \sum_{l=1}^{n} \frac{\partial p}{\partial x_l}(F(\hat{a}_i))\underbrace{(B\hat{a}_j - B\hat{a}_i)_l}_{\substack{B\hat{a}_j + b - (B\hat{a}_i + b) \\ = F(\hat{a}_j) - F(\hat{a}_i)}}$$

$$= \sum_{l=1}^{n} \frac{\partial p}{\partial x_l}(a_i)\cdot(a_j - a_i)$$

$$= \nabla p(a_i)\cdot(a_j - a_i) = \phi_{ij}(p)$$

$\Rightarrow$ Two Hermite n-simplices with above DoFs (derivatives along edges) are equivalent.

However, if we use partial derivatives instead for DoFs we <u>do not</u> get affine equivalence (Similarly, if use derivatives normal to edges)

Whenever two finite elements are affine equivalent use following correspondences:

$$\hat{x} \in \hat{T} \quad \longleftrightarrow \quad x = F(\hat{x}) \in T$$

$$\hat{p} \in \hat{P} \quad \longleftrightarrow \quad p = \hat{p} \circ F^{-1} \in P$$

We can write that

$$\hat{p}(\hat{x}) = p(x) \quad \forall \hat{x} \in \hat{T}, \hat{p} \in \hat{P}$$

<u>Theorem</u> Let $(\hat{T}, \hat{P}, \hat{\Sigma})$ and (T, P, Σ) be two affine-equivalent finite elements and let $\hat{p}_1, \ldots, \hat{p}_N$ be the basis functions of $(\hat{T}, \hat{P}, \hat{\Sigma})$. Then, the functions

$p_i := \hat{p}_i \circ F^{-1}$, $i = 1, \dots, N$ are the basis functions of the finite element (T, P, Σ). In addition, the $\hat{P}$-interpolation operators $\hat{\Pi}$ and the P-interpolation Π satisfy

$$\widehat{\Pi v} = \hat{\Pi}\hat{v} \quad \forall \hat{v} \in \mathcal{D}(\hat{\Pi})$$

$(\widehat{\Pi v} = (\Pi v) \circ F)$

<u>Proof</u> Let $\hat{\Sigma} = \{\hat{\phi}_i\}_{i=1}^N$ and denote $\phi_i(v) := \hat{\phi}_i(v \circ F)$.
Then, $\{\phi_i\}_{i=1}^N = \Sigma$. We have that

$$\phi_j(p_i) = \hat{\phi}_j(p_i \circ F) = \hat{\phi}_j(\hat{p}_i) = \delta_{ij}$$

$\Rightarrow p_1, \dots, p_N$ are basis functions of (T, P, Σ).

Furthermore,

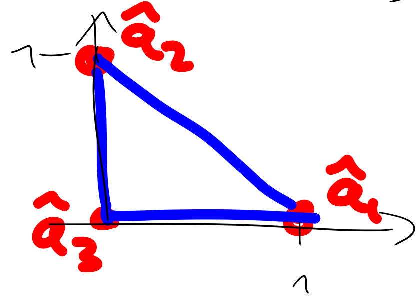
$$\hat{\Pi}\hat{v} = \sum_{i=1}^N \hat{\phi}_i(\hat{v})\hat{p}_i = \sum_{i=1}^N \hat{\phi}_i(v \circ F) p_i \circ F = \Big(\underbrace{\sum_{i=1}^N \phi_i(v) p_i}_{\Pi v}\Big) \circ F = \widehat{\Pi v}$$

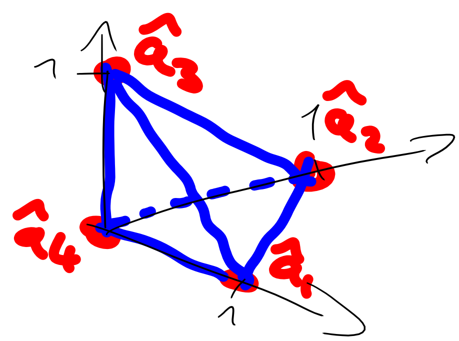
$\square$

System of finite elements is called an <u>affine-system</u> if all elements are affine-equivalent to a single finite element $(\hat{T}, \hat{P}, \hat{\Sigma})$, which does not need to be contained in the system, called the <u>reference finite element</u>.

If an affine system consists of n-simplices, the usual choice for $\hat{T}$ is the <u>unit n-simplex</u> with vertices

$$\hat{a}_1 = (1, 0, \dots, 0), \ \hat{a}_2 = (0, 1, 0, \dots, 0), \dots, \hat{a}_n = (0, 0, \dots, 0, 1),$$
$$\hat{a}_{n+1} = (0, \dots, 0)$$





Let T be any n-simplex with vertices $\underline{a}_1, \dots, \underline{a}_{n+1} \in \mathbb{R}^n$, $\underline{a}_j = (a_{ij})_{i=1}^n$, $j = 1, \dots, n+1$. According to the definition of an n-simplex, the matrix

$$A_T = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1,n+1} \\ a_{21} & a_{22} & & a_{2,n+1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{n,n+1} \\ 1 & 1 & & 1 \end{pmatrix} \text{ is non-singular}$$

$$\Rightarrow B_T = \begin{pmatrix} a_{11}-a_{1,n+1} & a_{12}-a_{1,n+1} & \cdots & a_{1n}-a_{1,n+1} \\ a_{21}-a_{2,n+1} & a_{22}-a_{2,n+1} & \cdots & a_{2n}-a_{2,n+1} \\ \vdots & \vdots & & \vdots \\ a_{n1}-a_{n,n+1} & a_{n2}-a_{n,n+1} & \cdots & a_{nn}-a_{n,n+1} \end{pmatrix} \text{ is non-singular,}$$

Define $F_T : \mathbb{R}^n \to \mathbb{R}^n$ by

$F_T(\hat{x}) = B_T \hat{x} + b_T$, $\hat{x} \in \mathbb{R}^n$, $b_T = (a_{1,n+1}, \dots, a_{n,n+1})$

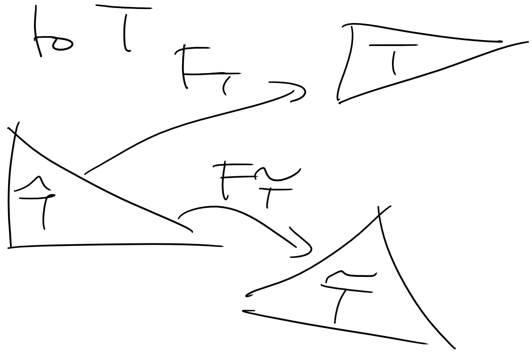
Then F_T is an invertible affine mapping and

$F_T(\hat{a}_i) = a_i$, $i = 1, \dots, n+1$. Since F_T is affine and the n-simplices are convex, we get $F_T(\hat{T}) = T$.

Note that $F_T^{-1}(x) = \begin{pmatrix} \lambda_1(x) \\ \vdots \\ \lambda_n(x) \end{pmatrix}$ (since $F_T^{-1}(a_i) = \hat{a}_i$)

Let $T, \tilde{T}$ be any n-simplices in $\mathbb{R}^n$. Then,

$F := F_T \circ F_{\tilde{T}}^{-1}$ is an invertible affine mapping which maps $\tilde{T}$ to T



On the reference n-simplex $\hat{T}$, the barycentric coordinates the form

$$\hat{\lambda}_i(\hat{x}) = \hat{x}_i, \quad i = 1, \ldots, n, \quad \hat{\lambda}_{n+1}(\hat{x}) = 1 - \sum_{i=1}^{n} \hat{x}_i$$

If an affine family consists of n-rectangles, the reference element is usually chosen as

$$\hat{T} = [0,1]^n \quad \text{or} \quad \hat{T} = [-1,1]^n.$$

Concept of affine families of finite element is fundamental important for two reasons:

- practical computations can be computed on reference element
- interpolation theory for finite element spaces which forms basis of convergence results.

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