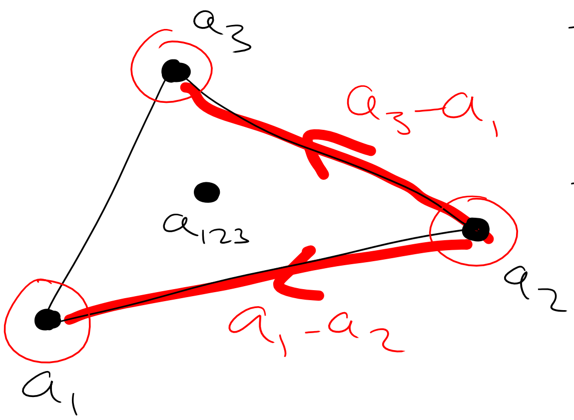
Hermite finite elements

Degrees of freedom are not only function value at some points but also values of derivatives in certain directions at some points

Example $T =$ triangle $P_T = P_3(T)$



- values at vertices & barycentre of element
- derivatives at vertices (either in coordinate direction or along edge)

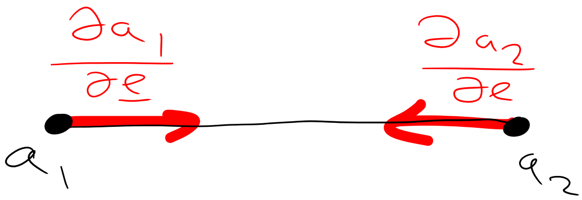
$$\Sigma_T = \{ p(a_1), p(a_2), p(a_3), p(a_{123}), \nabla p(a_1), \nabla p(a_2), \nabla p(a_3) \}$$

$$\Sigma_T' = \{ p(a_1), p(a_2), p(a_3), p(a_{123}), \nabla p(a_i) \cdot (a_j - a_i), j \neq i, i = 1, .., 3 \}$$

↳ Basis equivalent but different advantages/disadvantages

Degrees of freedom corresponding to an edge uniquely determine a cubic function on this edge

$$\Rightarrow X_h \subset C(\overline{\Omega}) \quad (\Rightarrow X_h \subset H^1(\Omega))$$



Can also define finite elements in such a way that

$$X_h \subset C^1(\overline{\Omega}) \Rightarrow X_h \subset H^2(\Omega);$$

for example, for biharmonic equation.

General definition of finite element

Definition A finite element in $\mathbb{R}^n$ is a triple (T, P, Σ) where

1) T is a bounded closed subset of $\mathbb{R}^n$ with non-empty interior and Lipschitz continuous boundary

2) P (or P_T) is a finite dimensional space of real functions on T; $N = \dim P$

3) Σ set of N linear form $\phi_i, i=1,\dots,N$, defined on P such that the set is P-unisolvent; i.e.,

$$\forall \alpha_1,\dots,\alpha_N \in \mathbb{R} \quad \exists!\, p \in P : \phi_i(p) = \alpha_i, i=1,\dots,N$$

$$\text{Unisolvence} \Rightarrow \begin{cases} \phi_1,\dots\phi_N \text{ are linearly independent} \\ \text{There } \exists \text{ functions } p_i \in P, i=1,\dots,N : \phi_j(p_i) = \delta_{ij}, i,j=1..N \end{cases}$$

$$\Rightarrow p = \sum_{i=1}^{N} \phi_i(p) p_i \quad \forall p \in P.$$

Linear forms $\phi_1 \dots \phi_N$ – degrees of freedom of the finite element

Functions $p_1, \dots, p_N$ – basis functions of the finite element

Remarks

- P-unisolvence of Σ is equivalent to fact that $\phi_1 \dots \phi_N$ form a basis in dual space of P
  $\Rightarrow$ basis $\{p_i\}_{i=1}^N$ and $\{\phi_i\}_{i=1}^N$ are dual basis in algebraic sense.
- To verify that Σ is P-unisolvent there are two basic ways:
  - Construct basis functions of P satisfying $\phi_j(p_i) = \delta_{ij}$
  - Show that $\phi_i(p) = 0, i=1\dots,N \Rightarrow p = 0$ for any $p \in P$

<u>Definition</u> Assume the functionals from Σ are defined on a larger space $Q \supset P$. We define a projection $\Pi : Q \to P$ by

$$\Pi v = \sum_{i=1}^{N} \phi_i(v) p_i , \quad v \in Q$$

Π is called the P-interpolation of function v

(sometime use Π_T for a specific finite element)

Since Σ is P-unisolvent the P-interpolation of v can be equivalently defined by

$$\Pi v \in P, \quad \phi_i(\Pi v) = \phi_i(v) , \quad i = 1, \dots, N.$$

<u>Remarks</u>

- If the linear forms in Σ represent function values the we speak about <u>Lagrange interpolation</u> ($\mathcal{D}(\Pi) = Q = C(T)$)
- If the forms in Σ represent both function values and values of derivatives, we speak about <u>Hermite interpolation</u> ($\mathcal{D}(\Pi) = Q = C^s(T), s > 0$)
- Π is a projection; i.e

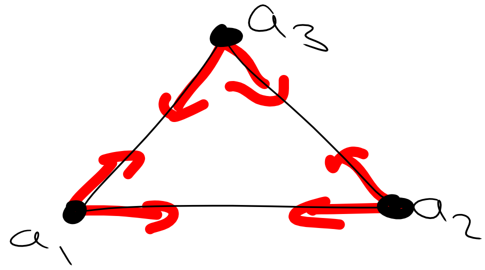
$$\Pi p = p \quad \forall p \in P.$$

<u>Definition</u> Two finite elements (T, P, Σ) and $(\tilde{T}, \tilde{P}, \tilde{\Sigma})$ are <u>equal</u> if $T = \tilde{T}$, $P = \tilde{P}$, and $\Pi_T = \Pi_{\tilde{T}}$.

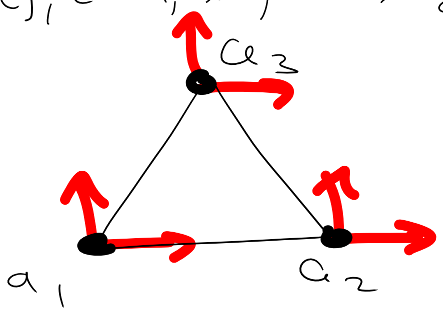
(For example, Hermite - different DoF but same projection).

<u>Example</u> - Reduced cubic Hermite n-simplex (Fig 8.2)

$$\Sigma = \{ p(a_i),\ i=1,\dots,n+1;\ \nabla p(a_i)\cdot(a_j - a_i),\ i=1,\dots,n+1 \}$$



$$\Sigma' = \{ p(a_i),\ i=1,\dots,n+1;\ \frac{\partial p}{\partial x_k}(a_i),\ i=1,\dots,n+1,\ k=1,\dots,n \}$$



Denote by Π and Π' the respective interpolation operators. Then, $\mathcal{D}(\Pi) = \mathcal{D}(\Pi') = C^1(T)$

For any function $v \in C^1(T)$ we have that

$$(\Pi v)(a_i) = v(a_i) \quad i=1,\dots,n+1$$

$$\nabla(\Pi v)(a_i)\cdot(a_j - a_i) = \nabla v(a_i)\cdot(a_j - a_i) \quad i,j=1,\dots,n+1$$

$$\Rightarrow \nabla(\Pi v)(a_i) = \nabla v(a_i) \quad i=1,\dots,n+1$$

since $\{a_j - a_i\}_{i\neq j}$ are linearly independent

$$(\Pi' v)(a_i) = v(a_i), \quad i=1,\dots,n+1$$

$$\frac{\partial(\Pi' v)}{\partial x_k}(a_i) = \frac{\partial v}{\partial x_k}(a_i), \quad i=1,\dots,n+1,\ k=1,\dots,n$$

$$\Rightarrow (\Pi v)(a_i) = (\Pi' v)(a_i), \quad i=1,\dots,n+1$$

$$\nabla(\Pi v)(a_i) = \nabla(\Pi' v)(a_i), \quad i=1,\dots,n+1$$

$\Rightarrow \Pi v = \Pi' v \Rightarrow \Pi \equiv \Pi'$ and (T,P,Σ) & (T,P,Σ') are equal.

Let (T, P, Σ) and $(\hat{T}, \hat{P}, \hat{\Sigma})$ are any two Lagrange n-simplices of order k; i.e,

$$T = \text{n-simplex}, \ P = P_k(T), \ \Sigma = \{p(z) : z \in L_k(T)\}$$

$$\hat{T} = \text{n-simplex}, \ \hat{P} = P_k(\hat{T}), \ \hat{\Sigma} = \{p(\hat{z}) : \hat{z} \in L_k(\hat{T})\}$$

There exists a mapping $F : \mathbb{R}^n \to \mathbb{R}^n$ satisfying

$$F(\hat{x}) = B\hat{x} + b, \ B \in \mathbb{R}^{n\times n}, \ b \in \mathbb{R}^n, \ B \text{ is non-singular} \quad (*)$$

and $F(\hat{T}) = T$ (will prove existence later).

A mapping satisfying $(*)$ is called an <u>invertible affine mapping</u>.

We observe that

$$\{p : T \to \mathbb{R} : p = \hat{p} \circ F^{-1}, \hat{p} \in \hat{P}\} \equiv P$$

$$\underline{\{p(F(\hat{z})) : \hat{z} \in L_k(\hat{T})\} \equiv \Sigma}$$

Denote $\Sigma = \{\phi_i\}_{i=1}^N$, $\hat{\Sigma} = \{\hat{\phi}_i\}_{i=1}^N$

This means that $\phi_i(p) = p(z_i) \qquad z_i \in L_k(T)$

$$\hat{\phi}_i(\hat{p}) = \hat{p}(\hat{z}_i) \qquad \hat{z}_i \in L_k(\hat{T})$$

For appropriate number of points, we have that $F(\hat{z}_i) = z_i$, $i = 1, \dots, N$. Thus,

$$\underline{\phi_i(p) = \hat{\phi}_i(p \circ F), \ i = 1 \dots N}$$

(equivalent)

Definition Two finite elements $(\hat{T}, \hat{P}, \hat{\Sigma})$ and (T, P, Σ) are affine-equivalent if there exists an invertible affine mapping $F: \mathbb{R}^n \to \mathbb{R}^n$ such that

$$T = F(\hat{T})$$

$$P = \{ p : T \to \mathbb{R} : p = \hat{p} \circ F^{-1}, \hat{p} \in \hat{P} \}$$

$$\Sigma = \{ \phi : \phi(p) = \hat{\phi}(p \circ F), \hat{\phi} \in \hat{\Sigma} \}$$

Remark If $(\hat{T}, \hat{P}, \hat{\Sigma})$ is a finite element and (T, P, Σ) is affine equivalent to this, then (T, P, Σ) is also a finite element. Can generate finite elements by simple using one fixed finite element $(\hat{T}, \hat{P}, \hat{\Sigma})$ called the reference element:

For example, for rectangle $(n=2)$ or order to define the reference rectangle $(\hat{T}, \hat{P}, \hat{\Sigma})$ ($\hat{T} = [0,1]^2$ for example) and using an affine mapping we can obtain a finite element for any T which is a parallelogram

We can also consider more general mappings F; e.g. for arbitrary convex quadrilaterals or elements with curved boundaries.

Remark To verify unisolvence of (T, P, Σ) sufficient to investigate an affine-equivalent finite element $(\hat{T}, \hat{P}, \hat{\Sigma})$, where $\hat{T}$ has a simple geometry.