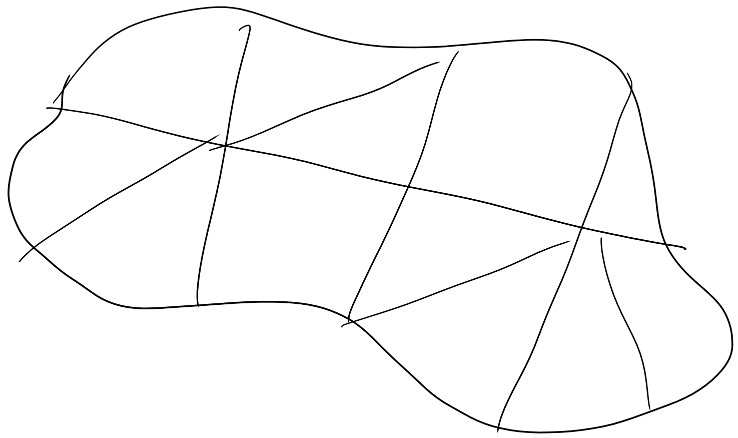
Finite Element Spaces

Finite element method is Galerkin method with special construction of spaces. - How do we construct them?

- First decompose domain Ω into subdomains



The set $\overline{\Omega}$ is decomposed into finitely many subsets T, which form a set $\mathcal{T}_h$ called the the triangulation (or mesh) of Ω, which is assumed to process the following properties:

($\mathcal{T}_h 1$) Each set $T \in \mathcal{T}_h$ is closed and its interior is non-empty and bounded

($\mathcal{T}_h 2$) The boundary ∂T of each $T \in \mathcal{T}_h$ is Lipschitz continuous

($\mathcal{T}_h 3$) $\overline{\Omega} = \bigcup_{T \in \mathcal{T}_h} T$

($\mathcal{T}_h 4$) The intersection of the interiors of any two different sets from $\mathcal{T}_h$ is empty.

- Construct functions on each subdomain and "join" to be subspace of H^1/H^2.

Let $\mathcal{T}_h$ be a triangulation of Ω. We want to construct a finite dimensional space approximating V.
Let X_h be an arbitrary finite dimensional space of functions on $\overline{\Omega}$ and denote $P_T = \{v_h|_T : v_h \in X_h\}$, $T \in \mathcal{T}_h$ be the restriction of X_h to T.

<u>Theorem</u> Let $P_T \subset H^1(T)$ $(\operatorname{int} T)$ $\forall T \in \mathcal{T}_h$ and $X_h \subset C(\overline{\Omega})$. Then, $X_h \subset H^1(\Omega)$ and
$X_{0h} := \{v_h \in X_h : v_h = 0 \text{ on } \partial\Omega\} \subset H^1_0(\Omega)$

<u>Proof</u> Consider any $v \in X_h$. Since $X_h \subset C(\overline{\Omega})$, it is clear that $v \in L^2(\Omega)$. It is sufficient to find $v_1, \dots, v_n \in L^2(\Omega)$ such that

$$\int_\Omega v \frac{\partial \varphi}{\partial x_i}\, dx = -\int_\Omega v_i \varphi\, dx \quad \forall \varphi \in C_0^\infty(\Omega)$$
$$i = 1, \dots, n$$

(weak derivatives in each comp.)

Let $v_i|_T := \dfrac{\partial v|_T}{\partial x_i}$. Then, $v_i \in L^2(\Omega)$

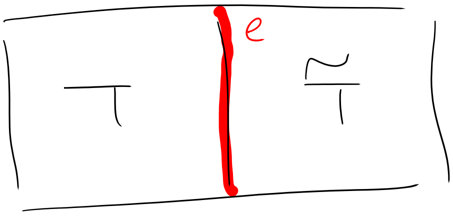
By Greens identity on T

$$\int_T v_i \varphi\, dx = \int_T \frac{\partial v}{\partial x_i} \varphi\, dx$$
$$= -\int_T v \frac{\partial \varphi}{\partial x_i}\, dx + \int_{\partial T} v \varphi n_{\partial T, i}\, ds$$

$$\Rightarrow \int_\Omega v_i \varphi\, dx = -\int_\Omega v \frac{\partial \varphi}{\partial x_i}\, dx + \underbrace{\sum_{T \in \mathcal{T}_h} \int_{\partial T} v \varphi n_{\partial T, i}\, ds}_{0}$$

$$\Rightarrow v \in H^1(\Omega)$$

Γ $v=0$ on $\partial\Omega$ & interior edges cancel:



– on shared edge $v\varphi$ same $(v\varphi|_T)|_e = (v\varphi|_{\tilde{T}})|_e$ but $n_{\partial T,i} = -n_{\partial\tilde{T},i}$

As $H_0^1(\Omega) := \{v \in H^1(\Omega) : v = 0 \text{ on } \partial\Omega\}$ then clearly $X_{0h} \subset H_0^1(\Omega)$ □

<u>Remark</u> Important result as shows can define subspace of $H^1(\Omega)$ by constructing locally on each T a subspace of $H^1(\Omega)$ and then ensuring continuity.

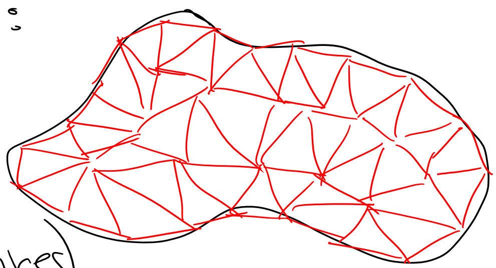
<u>Theorem</u> Let $P_T \subset H^2(\Omega)$ $\forall T \in \mathcal{T}_h$ and $X_h \in C^1(\bar{\Omega})$. Then, $X_h \subset H^2(\Omega)$, $X_{0h} \subset H^2(\Omega) \cap H_0^1(\Omega)$, and $X_{00h} = \{v_h \in X_h : v_h = \frac{\partial v_h}{\partial n} = 0 \text{ on } \partial\Omega\} \subset H_0^2(\Omega)$

More general variants of FEM:

1) can use more general variational problems; e.g. variational inequalities, mixed and hybrid formulations, boundary formulations (boundary EM $\to$ BEM)

2) space V_h is not subspace of V:

a) $\partial\Omega$ is curved $\Rightarrow \Omega$ is approximated



b) functions in V_h (or derivatives) may have discontinuities across interface between elements

3) bilinear form and linear functional are approximated (numerical quadrature)

4) the discrete problem can be enriched by additional terms (improves stability).

Pure Galerkin method = conforming finite element method

Finite element spaces on simplices

$P_k = \{ \sum_{|\alpha| \le k} \gamma_\alpha x^\alpha : \gamma_\alpha \in \mathbb{R} \}$ polynomial up to order $k \ge 0$

$(x \in \mathbb{R}^n, x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}, n \ge 1)$

$\dim P_k = \binom{n+k}{k}$

For any set $A \subset \mathbb{R}^n$ we denote by $P_k(A) = \{ p|_A : p \in P_k \}$

If A has non-empty interior, then $\dim P_k(A) = \dim P_k$.

Let $a_1, \dots, a_{n+1} \in \mathbb{R}^n$ ($n+1$ points), $a_j = (a_{ij})_{i=1}^n$ are such that

$$\mathbb{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1,n+1} \\ a_{21} & a_{22} & \cdots & a_{2,n+1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{n,n+1} \\ 1 & 1 & \cdots & 1 \end{pmatrix}$$

is invertible

($\Leftrightarrow$ the points are not contained in a hyperplane).

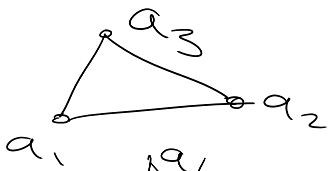
Let T be the convex hull of the points $a_1, \dots a_{n+1}$, i.e.,

$$T = \{ x = \sum_{j=1}^{n+1} \lambda_j a_j : \lambda_j \in [0,1], j = 1, \dots, n+1, \sum_{j=1}^{n+1} \lambda_j = 1 \}$$

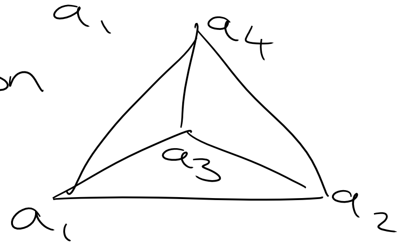
Then T is called an n-simplex in $\mathbb{R}^n$. The points $a_1, \dots, a_{n+1}$ are called the vertices of the n-simplex.

Ex $n=1$: T is the closed interval $[a_1, a_2]$

$n=2$: T is a triangle



$n=3$: T is a tetrahedron



For any $m \in \{0, 1, \dots, n-1\}$ an m-face of T is any m-simplex whose $m+1$ vertizes belong to the vertices of T.

$(n-1)$-face generally called 'face'

1-face generally called 'edge'

Barycentric coordinates $\lambda_j = \lambda_j(x)$, $j = 1, \dots, n+1$ of and $x \in \mathbb{R}^n$ with respect to vertizes $a_1, \dots, a_{n+1}$ of T are defined by

$$\sum_{j=1}^{n+1} \lambda_j a_j = x, \quad \sum_{j=1}^{n+1} \lambda_j = 1$$

Can be written as

$$\mathbb{A} \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \\ 1 \end{pmatrix}$$

As $\mathbb{A}$ is non-singular the barycentric coordinates always exist and are unique.

We have by explizit computation

$$\lambda_i = \sum_{j=1}^{n} b_{ij} x_j + b_{i\,n+1} \quad i = 1, \dots, n+1$$

where $\mathbb{B} = (b_{ij})_{i,j=1}^{n+1}$ is the inverse of $\mathbb{A}$. Thus,

$\lambda_i \in P_1$, $i = 1, \dots, n+1$, $\lambda_i(a_j) = \delta_{ij}$ $i,j = 1, \dots, n+1$.

These properties define the barycentric coordinates uniquely.

<u>Theorem</u> Let T be an n-simplex with vertices $a_1, \dots, a_{n+1}$. For any $k \geq 1$ denote by

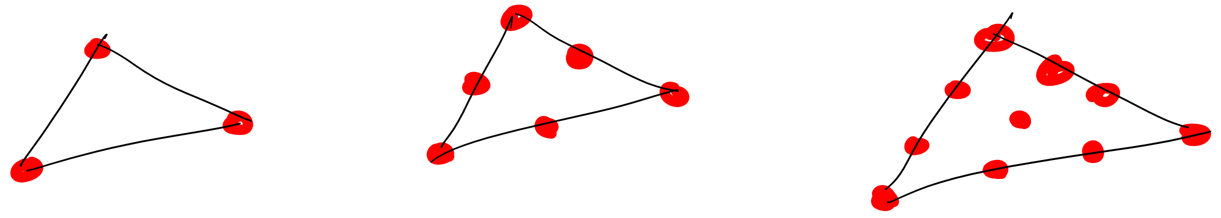
$$L_k(T) = \left\{ x = \sum_{j=1}^{n+1} \lambda_j a_j : \lambda_j \in \left\{0, \frac{1}{k}, \frac{2}{k}, \dots, \frac{k-1}{k}, 1\right\} \right.$$
$$\left. j = 1, \dots, n+1, \ \sum_{j=1}^{n+1} \lambda_j = 1 \right\}$$

Then,

1) $M_{n,k} := \operatorname{card} L_k(T) = \dim P_k$
2) If we denote by $z_1, \dots, z_{M_{n,k}}$ the elements of $L_k(T)$, then there is a basis $p_1, \dots, p_{M_{n,k}}$ of P_k satisfying $p_i(z_j) = \delta_{ij}$, $i, j = 1, \dots, M_{n,k}$.
3) Any polynomial $p \in P_k$ is uniquely determined by its values on the set $L_k(T)$ and $p = \sum_{i=1}^{M_{n,k}} p(z_i) p_i$.

$L_k(T)$ is called the principal lattice of order k of the n-simplex T.

<u>$k=1$</u> <u>$k=2$</u> <u>$k=3$</u>



<u>Proof</u>

① <u>$k=1$</u> $L_1(T) = \left\{ x = \sum_{j=1}^{n+1} \lambda_j a_j : \lambda_j = \{0,1\}, \sum_{j=1}^{n+1} \lambda_j = 1 \right\}$

$= \{a_1, \dots, a_{n+1}\}$

$\Rightarrow \operatorname{card} L_1(T) = n+1 = \dim P_1$

The basis functions are simply the barycentric coordinates $\lambda_1, \dots, \lambda_{n+1}$.

(2) $n=1$

$$L_k(T)=\{x=\lambda_1 a_1+\lambda_2 a_2,\ \lambda_1,\lambda_2\in\{0,\tfrac{1}{k},\dots,\tfrac{k-1}{k},1\},\ \lambda_1+\lambda_2=1\}$$

$$=\{x_j=\frac{j a_1+(k-j)a_2}{k},\ j=0,\dots,k\}$$

$\operatorname{card} L_k(T)=k+1=\dim P_k$ (in 1D)

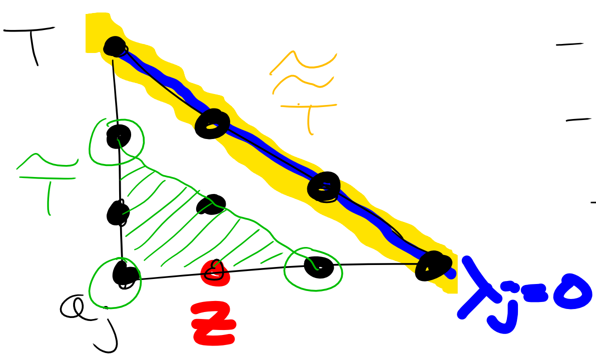
Since $\lambda_1(x_j)=\frac{j}{k}$, $p_i(x)=\frac{1}{\alpha_i}\prod_{\substack{j=0\\ j\neq i}}^{k}\left(\lambda_1(x)-\frac{j}{k}\right)$

- Clearly
  - p_i polynomial of degree k
  - $p_i(x_j)=0 \quad i\neq j$
  - $p_i(x_i)\neq 0$

Define $\alpha_i=\prod_{\substack{j=0\\ j\neq i}}^{k}\frac{i-j}{k}$ then $p_i(x_i)=1$.

(3) Let $n>1$ & $k>1$. We shall prove that for any $z\in L_k(T)$ $\exists$ n-simplex $\tilde{T}$ and $\tilde{p}\in P_1$ such that $z\in L_{k-1}(\tilde{T})\subset L_k(T)$, $\tilde{p}(z)=1$, $\tilde{p}(y)=0$ $\forall y\in L_k(T)\setminus L_k(\tilde{T})$

Ex. $k=3$



- $z\in L_{k-1}(\tilde{T})\subset L_k(T)$
- $\lambda_j(y)=0$ $\forall y\in L_k(T)\setminus L_{k-1}(\tilde{T})$
- $\lambda_j(z)\neq 0$
- So set $\tilde{p}=c\lambda_j$, $c>0$, such that $\tilde{p}(z)=1$

$\Rightarrow$ all properties found $\lrcorner$

We know that $\lambda_j(z) \neq 0$ for some $j \in \{1, .., n+1\}$

Let $j = n+1$ (w.l.o.g.). Denote $\tilde{L}_k(T) = \{x \in L_k(T) : \lambda_{n+1}(x) \neq 0\}$

$$x \in \tilde{L}_k(T) \iff x = \sum_{j=1}^{n+1} \lambda_j a_j, \quad \sum_{j=1}^{n+1} \lambda_j = 1,$$

$$\lambda_j \in \{0, \tfrac{1}{k}, \dots, \tfrac{k-2}{k}, \tfrac{k-1}{k}\}, j = 1, \dots, n$$

$$\lambda_{n+1} \in \{\tfrac{1}{k}, \dots, \tfrac{k-1}{k}, 1\}$$

$$\left[\frac{k\lambda_j}{k-1} \in \{0, \tfrac{1}{k-1}, \dots, \tfrac{k-2}{k-1}, 1\} =: \tilde{\lambda}_j \right.$$

$$\left. \frac{k\lambda_{n+1} - 1}{k-1} \in \{0, \tfrac{1}{k-1}, \dots, \tfrac{k-2}{k-1}, 1\} =: \tilde{\lambda}_{n+1} \right]$$

$$\iff x = \sum_{j=1}^{n+1} \frac{k-1}{k} \tilde{\lambda}_j a_j + \frac{a_{n+1}}{k}, \quad \sum_{j=1}^{n+1} \tilde{\lambda}_j = 1$$

$$\left[\sum_{j=1}^{n+1} \tilde{\lambda}_j = \frac{k}{k-1} - \frac{1}{k-1} = 1 \right]$$

$$\iff x = \sum_{j=1}^{n+1} \tilde{\lambda}_j \boxed{\frac{(k-1)a_j + a_{n+1}}{k}} =: \tilde{a}_j \in L_k(T)$$

Define n-simplex using $\tilde{a}_j$ $(\tilde{T})$ $\hookrightarrow$ vertices of $\tilde{T}$

$$\Rightarrow \tilde{L}_k(T) = L_{k-1}(\tilde{T})$$

$$\tilde{p} = \frac{\lambda_{n+1}}{\lambda_{n+1}(z)}$$

$$L_k(T) \setminus L_{k-1}(\tilde{T}) = L_k(\tilde{\tilde{T}})$$

where $\tilde{\tilde{T}}$ is a $(n-1)$-simplex defined by $a_1, \dots, a_n$

(4) Let $z \in L_k(T)$, $T_k = T$

According to (3) there exists a sequence of n-simplices $T_{k-1}, \dots, T_1$ and functions $p_{k-1}, \dots, p_1 \in P$ so that for $i = 1, \dots, k-1$ $z \in L_i(T_i) \subset L_{i+1}(T_{i+1})$, $p_i(z) = 1$, $p_i(y) = 0$ $\forall y \in L_{i+1}(T_{i+1}) \setminus L_i(T_i)$.

Let $p_0 \in P_1$ such that $p_0(z) = 1$ and $p_0(y) = 0$ $\forall y \in L_1(T_1) \setminus \{z\}$. Set $p = \prod_{i=0}^{k-1} p_i$; then, $p \in P_k$, $p(z) = 1$ and $p(y) = 0$ $\forall y \in L_k(T) \setminus \{z\}$

(5) We know that $L_k(T) = L_{k-1}(\tilde{T}) \cup L_k(\tilde{\tilde{T}})$

$\tilde{T}$ - n-simplex, $\tilde{\tilde{T}}$ - $(n-1)$-simplex $\tilde{T} \cap \tilde{\tilde{T}} = \emptyset$

$$\Rightarrow M_{n,k} = M_{n-1,k} + M_{n,k-1}$$
$$= M_{n-1,k} + M_{n-1,k-1} + M_{n,k-2}$$
$$\vdots$$
$$= M_{n,1} + \sum_{j=2}^{n} M_{n-1,j}$$

Need to show that $M_{n,k} = \binom{n+k}{n}$

By induction $M_{1,k} = k+1 = \binom{k+1}{n}$;

then, assume holds for n & prove for $n+1$:

$$M_{n+1,k} = M_{n+1,1} + \sum_{j=2}^{k} M_{n,j} \qquad (M_{n,1} = n+1)$$
$$= n+2 + \sum_{j=2}^{k} \binom{n+j}{n}$$
$$= \sum_{j=0}^{k} \binom{n+j}{n}$$
$$= \binom{n+k+1}{n+1}$$

$\square$