Finite Element Methods 1

- Website: https://www.karlin.mff.cuni.cz/~congreve/teaching.php?c=WS2023_FEM1
- Exam 30 minute oral examination on topics from lectures
- Practical course credit:
  - 4 homeworks during semester
  - Credit if achieve 50% or more

Sobolev Spaces

Needed for FEM, recap, more detailed in diff. eqns. course

- $\Omega \subset \mathbb{R}^n, n \geq 1$ - bounded domain (bounded, open & connected set)
- $C(\Omega)$ - set of real continuous functions in Ω
- $C^k(\Omega), k \in \mathbb{N}$ - subset of $C(\Omega)$ having continuous derivative upto order k.
- For $v \in C^k(\Omega)$, define multi-index $\alpha = (\alpha_1, \ldots, \alpha_n), \alpha_i \in \mathbb{N}_0$, $|\alpha| := \alpha_1 + \ldots + \alpha_d \leq k$

$$D^\alpha v := \frac{\partial^{|\alpha|} v}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}}$$

- $C^\infty(\Omega)$ - set of infinitely smooth functions (i.e. belong to $C^k(\Omega)$ $\forall k \in \mathbb{N}$)
- $C_0^\infty(\Omega) = \{ v \in C^\infty(\Omega) : \operatorname{supp} v \subset \Omega \}$

$$\operatorname{supp} v = \overline{\{ x \in \Omega, v(x) \neq 0 \}}$$

  - set of infinitely smooth functions with compact support.
- Define $L^p(\Omega)$ with $p \in [1, \infty)$ (Lebesgue Space)

$$L^p(\Omega) := \left\{ v : \Omega \to \mathbb{R} : v \text{ measurable}, \int_\Omega |v(x)|^p \, dx < \infty \right\}$$

→ Banach space with norm

$$\|v\|_{L^p(\Omega)} = \left(\int_\Omega |v(x)|^p dx \right)^{1/p}$$

→ $p=2$ forms Hilbert space with norm

$$(u,v) = \int_\Omega u(x)\, v(x)\, dx$$

→ $p=\infty$ Banach space

$$L^\infty(\Omega) := \{ v : \Omega \to \mathbb{R} : v \text{ measurable}, \operatorname*{ess\,sup}_{x\in\Omega} |v(x)| < \infty \}$$

$$\|v\|_{L^\infty(\Omega)} = \operatorname*{ess\,sup}_{x\in\Omega} |v(x)|$$

- Space of locally integrable functions

$$L^{1,loc}(\Omega) := \{ v : \Omega \to \mathbb{R} : v \text{ measurable}, \int_K |v(x)| dx \; \forall K \subset \Omega \text{ compact} \}$$

$L^p(\Omega) \subset L^{1,loc}(\Omega)$ for any $p \in [1,\infty)$.

- Functions from $L^{1,loc}(\Omega)$ do not possess classical derivatives in general; however, interpreting them as distributions can define derivatives of arbitrary order.
- If a derivative can be identified with a function from $L^{1,loc}(\Omega)$ we call it the weak derivative; i.e, given multiindex α, function $v_\alpha \in L^{1,loc}(\Omega)$ is the α^{th}-weak derivative of $v \in L^{1,loc}(\Omega)$ if

$$\int_\Omega v_\alpha \varphi\, dx = (-1)^{|\alpha|} \int_\Omega v\, D^\alpha \varphi\, dx \quad \forall \varphi \in C_0^\infty(\Omega)$$

If weak derivative exists they are determined uniquely (as element of $L^{1,loc}(\Omega)$). Note, function $v \in C^k(\Omega)$, $k \in \mathbb{N}$ satisfies $\int_\Omega (D^\alpha v) \varphi\, dx = (-1)^{|\alpha|} \int_\Omega v\, D^\alpha \varphi\, dx \quad \forall \varphi \in C_0^\infty(\Omega)$

⇒ when classical derivatives exist they coincide with the weak derivatives.

- Will use $D^\alpha v$ to denote weak derivatives.

Definition (Sobolev Space)

Let $p \in [1,\infty]$, $k \in \mathbb{N}_0$, define

$$W^{k,p}(\Omega) := \{u \in L^p(\Omega) : D^\alpha u \in L^p(\Omega) \ \forall |\alpha| \le k\};$$

i.e, set of functions from $L^p(\Omega)$ whose weak derivatives upto k exist and belong to $L^p(\Omega)$.

Lemma Let $p \in [1,\infty]$, $k \in \mathbb{N}_0$. Then, $W^{k,p}(\Omega)$ is a Banach space with respect to the norm

$$\|u\|_{k,p,\Omega} := \begin{cases} \left(\sum_{|\alpha|\le k} \int_\Omega |D^\alpha u|^p dx\right)^{1/p} = \left(\sum_{|\alpha|\le k} \|D^\alpha u\|^p_{L^p(\Omega)}\right)^{1/p}, & p \in [1,\infty) \\ \max_{|\alpha|\le k} \|D^\alpha u\|_{L^\infty(\Omega)}, & p = \infty \end{cases}$$

Furthermore, we define semi-norms

$$|u|_{k,p,\Omega} := \begin{cases} \left(\sum_{|\alpha|=k} \|D^\alpha u\|^p_{L^p(\Omega)}\right)^{1/p}, & p \in [1,\infty) \\ \max_{|\alpha|=k} \|D^\alpha u\|_{L^\infty(\Omega)}, & p = \infty \end{cases}$$

Note that, $W^{k,p}(\Omega)$ is separable and for $p \in (1,\infty)$ reflexive. Also $W^{0,p}(\Omega) = L^p(\Omega)$, $\|\cdot\|_{0,p,\Omega} = \|\cdot\|_{L^p(\Omega)}$

For $p=2$ $W^{k,p}(\Omega)$ is a Hilbert space with inner product

$$(u,v)_{k,\Omega} = \sum_{|\alpha|\le k} (D^\alpha u, D^\alpha v)$$

Definition $W_0^{k,p}(\Omega) = \overline{C_0^\infty(\Omega)}^{\|\cdot\|_{k,p,\Omega}}$

(closure in norm $W^{k,p}(\Omega)$ -subspace)

$[W^{k,p}(\Omega)]'$ space of continuous linear functionals on $W^{k,p}(\Omega)$

(dual space)

For $p=2$ we use notation:

$H^k(\Omega) = W^{k,2}(\Omega)$, $H_0^k(\Omega) = W_0^{k,2}(\Omega)$, $\|\cdot\|_{k,\Omega} = \|\cdot\|_{k,2,\Omega}$, $|u|_{k,\Omega} = |u|_{k,2,\Omega}$

Lemma Let Ω be bounded domain; then, for $u \in W_0^{k,p}(\Omega)$

$$\int_\Omega |D^\beta u|^p dx \leq C \sum_{|\alpha|=k} \int_\Omega |D^\alpha u|^p dx \quad \forall |\beta| \leq k$$

where C constant depending only on Ω, p and k.

Friedrichs inequality

$$\|u\|_{k,p,\Omega} \leq C |u|_{k,p,\Omega} \quad \forall u \in W_0^{k,p}(\Omega)$$

Consequence Let Ω be a bounded domain; then, $|\cdot|_{k,p,\Omega}$ is a norm in $W_0^{k,p}(\Omega)$. For $p=2$

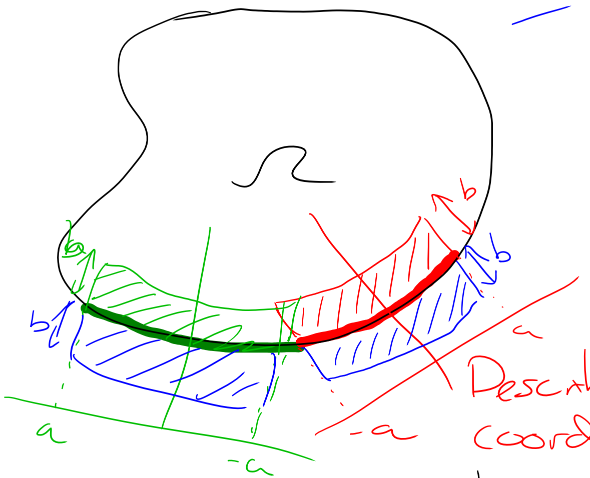
$$(u,v)_{H_0^k(\Omega)} = \sum_{|\alpha|=k} \int_\Omega D^\alpha u D^\alpha v \, dx$$

is an inner product in $H_0^k(\Omega)$.

In $H_0^1(\Omega)$: $(u,v)_{H_0^1(\Omega)} = \int_\Omega \nabla u \cdot \nabla v \, dx = (\nabla u, \nabla v)$ induces $|\cdot|_{1,\Omega}$

- Friedrichs inequality holds for any measurable domain $\Omega \subset \mathbb{R}^n$; however, other properties require assumptions on regularity of the boundary.

Boundary $\partial\Omega \in C^{k,1}$



- bounded domain
- assume representable by a finite number of local coordinate systems

Describe boundary as function on this coordinate system

— Then cover whole boundary.

$C^{k,1}$-regularity of the functions in the local coordinate systems (k derivatives — Lipschitz continuous)

Following properties hold for all coordinate systems

- Limits in local coordinate systems $(-a,a)$

- Strip above domain inside Ω (width b)
- Strip below domain outside Ω (width b)

Definition Bounded domain $\Omega \subset \mathbb{R}^n$ $(n \geq 2)$ is of class $C^{k,1}$ (with $k \in \mathbb{N}_0$) if there exists a finite number of local coordinate systems $S_1, \dots, S_M$, function $f_1, \dots, f_M$, and numbers $a, b > 0$ such that

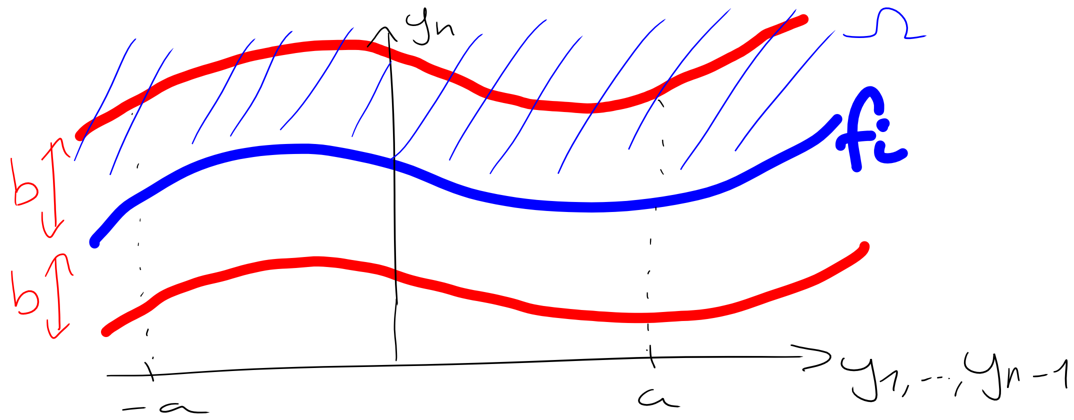
a) functions $f_1, \dots, f_M$ are k-times differentiable and have Lipschitz-continuous derivative of order k on closure of the set

$$K_{n-1} = \{ y = (y_1, \dots, y_{d-1}) : |y_j| < a,\ j = 1, \dots, n-1 \};$$

b) $\forall x \in \partial\Omega\ \exists i \in \{1, \dots, M\}$ and $y \in K_{n-1}$ such that $x = (y, f_i(y))$ in local coordinate system S_i,

c) in each local coordinate system S_i, $i = 1, \dots, M$

$$(y, y_n) \in \Omega \text{ if } y \in \overline{K_{n-1}},\ f_i(y) < y_n < f_i(y) + b$$
$$(y, y_n) \notin \Omega \text{ if } y \in \overline{K_{n-1}},\ f_i(y) - b < y_n < f_i(y)$$

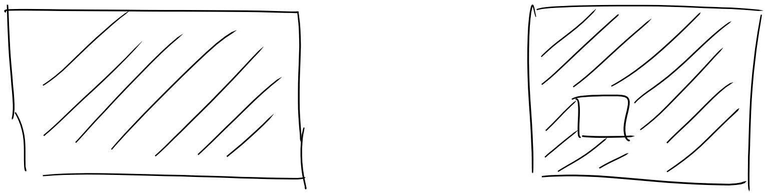


Remark If Ω is of class $C^{0,1}$ we say Ω is a Lipschitz continuous boundary.

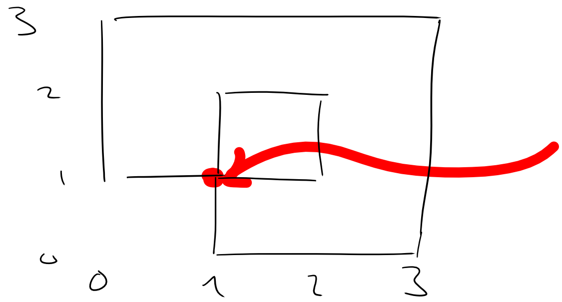
Examples

- Bounded convex domains in $\mathbb{R}^n$ have Lipschitz-continuous boundaries

- Bounded polygonal domains in $\mathbb{R}^2$ has Lipschitz-continuous boundaries if the boundary represents a simple curve

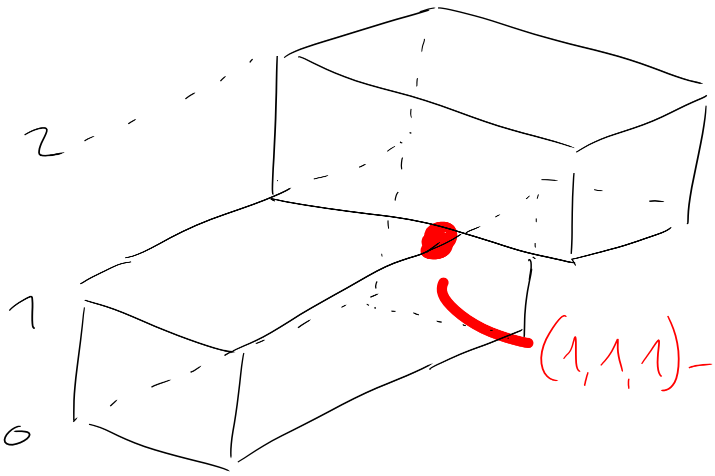


- Counterexample: $(0,3)^2 \setminus \{[1,2]^2 \cup [0,1]^2\}$



Not Lipschitz-continuous here

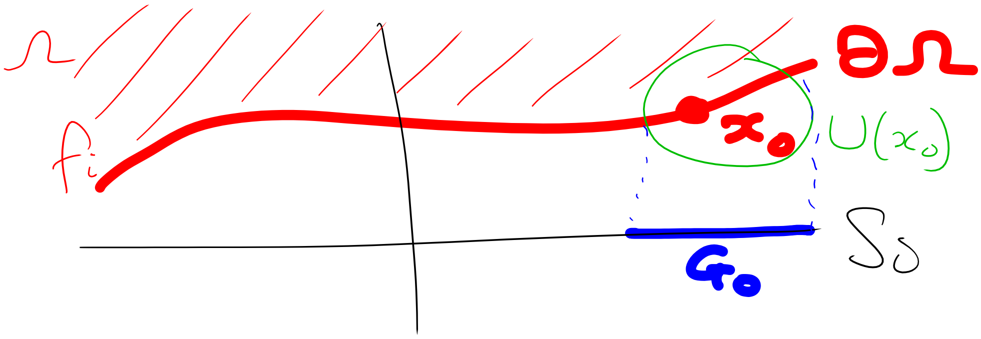
- Bounded polygonal domain in $\mathbb{R}^3$ does not, in general, have Lipschitz-continuous boundary i.e.g. interior of $[0,1]\times[0,2]\times[0,1] \cup [0,2]\times[1,2]\times[1,2]$



(1,1,1) - Not Lipschitz continuous

Surface Measure

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with Lipschitz continuous boundary. For every point $x_0 \in \partial\Omega$ there exists a neighbourhood $U(x)$ such that the are $U(x_0) \cap \partial\Omega$ is contained in the graph of the function f_i in some coordinate system S_i



Therefore, we define the $(n-1)$-dimensional Lebesgue surface measure of this area by

$$\sigma_{n-1}\left(U(x_0)\cap\partial\Omega\right)=\int_{G_0}\left(1+\left(\frac{\partial f_i}{\partial y_1}\right)^2+\dots+\left(\frac{\partial f_i}{\partial y_{n-1}}\right)^2\right)^{1/2}dy_{n-1}\dots dy_1$$

where $G_0\subset K_{n-1}$ is the projection of the surface onto the place $y_1,\dots,y_{n-1}$; i.e. $U(x_0)\cap\partial\Omega=\{(y,f_i(y)),y\in G_0\}$

In this way, it is possible to define measurable subdomains of the boundary $\partial\Omega$ and measurable & integrable functions on $\partial\Omega$. It can be shown that the surface measure does not depend on the choice of local coordinate systems.

<u>Gauss Integral Theorem</u> Let Ω be a bounded domain with Lipschitz boundary and $u\in C^1(\Omega)\cap C(\overline{\Omega})$; then,

$$\int_\Omega\frac{\partial u}{\partial x_i}dx=\int_{\partial\Omega}u\cdot n_i\,ds\qquad i=1,\dots,n$$

<u>L^p spaces on $\partial\Omega$</u> - defined analogously to on Ω; i.e, for $p\in[1,\infty)$ define

$$L^p(\partial\Omega)=\left\{u\text{ measurable}:\left(\int_{\partial\Omega}|u|^p ds\right)^{1/p}<\infty\right\}$$

Moreover, $L^p(\partial\Omega)$ is a Banach space with norm

$$\|u\|_{0,p,\partial\Omega}=\|u\|_{L^p(\partial\Omega)}=\left(\int_\Omega|u|^p ds\right)^{1/p}$$

Sobolev Embedding Theorem Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with Lipschitz continuous boundary. Consider any $k, j \in \mathbb{N}_0$ and $p, r \in [1, \infty)$; then,

$W^{k,p}(\Omega) \hookrightarrow W^{j,r}(\Omega)$ if $0 \leq j \leq k$ & $\frac{1}{p} - \frac{k-j}{n} \leq \frac{1}{r}$

$W^{k,p}(\Omega) \hookrightarrow C^{j}(\Omega)$ if $\frac{1}{p} - \frac{k-j}{n} < 0$

(continuous embedding)

$\left[W^{k,p}(\Omega) \subset W^{j,r}(\Omega) \quad \|v\|_{j,r,\Omega} \leq C \|v\|_{k,p,\Omega} \quad \forall v \in W^{k,p}(\Omega) \right]$

Remark $H^1(\Omega) \hookrightarrow L^r(\Omega)$ when?

$v \in H^1(\Omega) \Rightarrow v \in L^2(\Omega)$, $r \in (0,2)$

$$\left(\int_\Omega 1 \cdot |v|^r dx \right)^{1/r} \leq \left(\left(\int_\Omega |v|^{r \cdot \frac{2}{r}} \right)^{r/2} \left(\int_\Omega 1 \right)^{1-\frac{r}{2}} \right)^{1/r}$$

$$= \|v\|_{0,\Omega} |\Omega|^{1/r - 1/2}$$

Condition:

$$\frac{1}{2} - \frac{1-0}{n} \leq \frac{1}{r} < 1 \begin{cases} \text{If } n=2 \Rightarrow \text{holds for any } r>0 \\ \text{If } n=3 \Rightarrow \text{holds for } r \in (0,6) \end{cases}$$

Theorem (Rellich-Kondrashov)

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with Lipschitz continuous boundary. Then

$W^{k,p}(\Omega) \hookrightarrow\hookrightarrow W^{k-1,p}(\Omega) \quad \forall k \in \mathbb{N}, p \in [1, \infty)$

(compactly embedded)

i.e. sequence in $W^{k,p}$ bounded $\Rightarrow$ convergent subsequence in $W^{k-1,p}(\Omega)$. Seq. bounded in $H^1(\Omega) \Rightarrow$ conv. subseq. in $L^2(\Omega)$.

<u>Trace Theorem</u> Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with Lipschitz continuous boundary. Let $p \geq 1$, $kp < n$ and $\frac{1}{q} = \frac{1}{p} - \frac{kp-1}{p(n-1)}$. Then, there exists a unique continuous linear mapping $\gamma : W^{k,p}(\Omega) \to L^q(\partial\Omega)$ such that

$$\gamma u = u|_{\partial\Omega} \quad \forall u \in C^\infty(\bar{\Omega}).$$

If $kp = n$ then $\gamma \in \mathcal{L}(W^{k,p}(\Omega), L^q(\partial\Omega))$ $\forall q \geq 1$. Check!!

<u>Remark</u> Continuity of γ implies that

$$\|\gamma u\|_{0,q,\partial\Omega} \leq C \|u\|_{k,p,\Omega} \quad \forall u \in W^{k,p}(\Omega)$$

- $H^1(\Omega)$: $n=2$, $q \in [1,\infty)$; $n=3$, $q \in [1,4]$

upper decreases, but $\geq 2 \Rightarrow H^1(\Omega) \to L^2(\partial\Omega)$ always

$$\|u\|_{0,\partial\Omega} \leq C \|u\|_{1,\Omega}$$

<u>Lemma</u> Let Ω be a bounded domain, $u \in W^{1,p}(\Omega)$, $1 < p < \infty$.

Then, $u \in W_0^{1,p}(\Omega) \Leftrightarrow u|_{\partial\Omega} = 0$;

i.e, $W_0^{1,p}(\Omega)$ space of functions with zero trace.

<u>Theorem (density)</u> Let Ω be bounded domain with Lipschitz continuous boundary. Let

$$R(\Omega) := \{u|_\Omega, u \in C_0^\infty(\mathbb{R}^n)\}$$

Then, $R(\Omega)$ is dense in $W^{k,p}(\Omega)$ for $k \geq 0$ & $p \in [1,\infty)$.

<u>Prove Gauss Integral Theorem in $H^1(\Omega)$</u>

Consider any $u \in H^1(\Omega)$. Know infinitely smooth functions are dense in $H^1(\Omega)$. Then, $\exists \{u_k\}_{k=1}^\infty \subset C^\infty(\bar{\Omega})$ such that $u_k \to u$ in $H^1(\Omega)$; i.e, $\|u - u_k\|_{1,\Omega} \to 0$.

$$\Rightarrow \int_\Omega \frac{\partial u_k}{\partial x_i} dx = \int_{\partial\Omega} u_k n_i \, ds \qquad i = 1, \dots, n$$

$$\left|\int_\Omega \frac{\partial u}{\partial x_i}dx - \int_{\partial\Omega} u\cdot n_i\, ds\right| = \left|\int_\Omega \frac{\partial u}{\partial x_i} - \frac{\partial u_k}{\partial x_i}dx - \int_{\partial\Omega}(u-u_k)\cdot n_i\right| ds$$

$$\leq \int_\Omega \left|\frac{\partial}{\partial x_i}(u-u_k)\right|dx + \int_{\partial\Omega}|u-u_k|ds \quad (|n_i|\leq 1)$$

$$\leq \sqrt{|\Omega|}\,|u-u_k|_{1,\Omega} + \sqrt{|\partial\Omega|}\,|u-u_k|_{0,\partial\Omega}$$

$$\leq \sqrt{|\Omega|}\underbrace{\|u-u_k\|_{1,\Omega}}_{\to 0} + \sqrt{|\partial\Omega|}\,C\underbrace{\|u-u_k\|_{1,\Omega}}_{\to 0} \longrightarrow 0 \text{ as } k\to 0$$

$\square$

Lemma Let Ω be a bounded domain with Lipschitz continuous boundary, $G\subset\Omega$ subset with positive measure, $\Gamma\subset\partial\Omega$ subset of boundary with positive surface measure. Then, there exists constants $C_1 = C_1(n,p,G,\Omega)$ and $C_2=C_2(n,p,\Gamma,\Omega)$ such that for any $u\in W^{1,p}(\Omega)$, $p\geq 1$,

$$\|u\|_{0,p,\Omega} \leq C_1\left(|u|_{1,p,\Omega} + \left|\int_G u\,dx\right|\right)$$

$(p=2,\ G=\Omega \Rightarrow$ Poincaré inequality$)$

$$\|u\|_{0,p,\Omega} \leq C_2\left(|u|_{1,p,\Omega} + \left|\int_\Gamma u\,ds\right|\right) \quad \text{(Friedrich's inequality)}$$

Definition Let $k\geq 0$, $p\in(1,\infty)$, we denote by $W^{-k,q}(\Omega)$ the dual space of the Sobolev space $W_0^{k,p}(\Omega)$, $\frac{1}{p}+\frac{1}{q}=1$.

For $p=q=2$ $\quad H^{-k}(\Omega) = (H_0^k(\Omega))'$.

Prove of Poincaré

Prove $$\|u\|_{0,p,\Omega} \leq C\left(|u|_{1,p,\Omega} + \left|\int_G u\,dx\right|\right) \quad \forall u\in W^{1,p}(\Omega)$$

$G\subset\Omega$, $\mathrm{meas}_n(G)>0$ (positive measure)

Proof by contradiction, assume

$\forall C>0 \quad \exists u\in W^{1,p}(\Omega)$ such that

$$\|u\|_{0,p,\Omega} > C\left(|u|_{1,p,\Omega} + \left|\int_G u\,dx\right|\right)$$

Construct sequence related to C (integers) ($C=n$)

$$\{u_n\}_{n=1}^{\infty} \subset W^{1,p}(\Omega) : \|u_n\|_{0,p,\Omega} \geq n\left(|u_n|_{1,p,\Omega} + \left|\int_G u_n dx\right|\right)$$

Can multiple inequality by any positive constant without changing validity of inequality

So scale such that

$$\|u_n\|_{0,p,\Omega} = 1 \quad (\text{sequence bounded in } W^{1,p}(\Omega))$$

By Rellich $\quad W^{1,p}(\Omega) \hookrightarrow\hookrightarrow L^p(\Omega)$

$\Rightarrow \exists \{u_{n_k}\} \subset \{u_n\}$ such that $u_{n_k} \to u \in L^p(\Omega)$

(exists convergent subsequence in $L^p(\Omega)$);

hence $\{u_{n_k}\}$ is a Cauchy sequence in $L^p(\Omega)$

$$|u_n|_{1,p,\Omega} + \left|\int_G u_n dx\right| < \frac{1}{n} \quad (\text{as } \|u_n\|_{0,p,\Omega} \leq \|u_n\|_{1,p,\Omega} = 1)$$

$\Rightarrow |u_n|_{1,p,\Omega} \to 0 \quad \Rightarrow \{u_n\}$ is Cauchy sequence with respect to $|\cdot|_{1,p,\Omega}$

$$\|v\|_{1,p,\Omega} = \left(\|v\|_{0,p,\Omega}^p + |v|_{0,p,\Omega}^p\right)^{1/p} \quad \Rightarrow \text{Cauchy seq. w.r.t } \|\cdot\|_{1,p,\Omega}$$

Cauchy seq w.r.t this $\qquad$ Cauchy seq. w.r.t this

$\Rightarrow \{u_{n_k}\}$ is a Cauchy sequence in $W^{1,p}(\Omega)$

and as Banach sequence this sequence must converge to u (same limit as in $L^p(\Omega)$)

$\Rightarrow u_{n_k} \to u \quad$ in $W^{1,p}(\Omega)$

$\Rightarrow \|u\|_{1,p,\Omega} = 1 \quad$ (as norm of every element in seq $=1$)

$\quad |u|_{1,p,\Omega} = 0 \quad$ (as seminorm of $u_{n_k} \to 0$)

$\Rightarrow u = \text{constant}$ as first weak derivative vanish

Now consider $\int_G$ terms.

$$\left|\int_G u_n \, dx\right| \xrightarrow[(n\to\infty)]{} 0 \implies \int_G u \, dx = 0$$

$$\left(\left|\int_G u \, dx - \int u_n dx\right| = \left(\int_G u - u_n \, dx\right) \leq \int_G 1 \cdot |u - u_n| \, dx \right.$$

$$\left. \leq \underbrace{\left(\int_G |u-u_n|^p\right)^{1/p}}_{\|u-u_n\|_{0,p,\Omega} \to 0} \left(\int_G 1\right)^{1-1/p} \longrightarrow 0 \right)$$

$\int_G u \, dx = 0 \longrightarrow$ but u is constant & G positive measure $\implies u = 0$

which contradicts $\|u\|_{1,p,\Omega} = 1$. $\square$