27.11.2023 — Homework 3

Finite Element Methods 1

Due date: 11th December 2023

Submit a PDF/scan of the answers to the following questions before the deadline via the *Study Group Roster* (*Záznamník učitele*) in SIS, or hand-in directly at the practical class on the 11th December 2023.

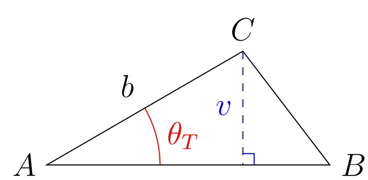
1. (1 point) Consider a triangulation $\mathcal{T}_h$ of $\Omega \subset \mathbb{R}^2$ consisting of simplices T with diameter h_T, and define by ϱ_T the diameter of the largest inscribed ball in T. Show that the condition

$$\frac{h_T}{\varrho_T} \leq \sigma, \qquad \text{for all } T \in \mathcal{T}_h, \tag{1.1}$$

where the constant σ is independent of T, is equivalent to the condition that all angles in all $T \in \mathcal{T}_h$ are bounded from below by a positive constant θ_0 independent of T.

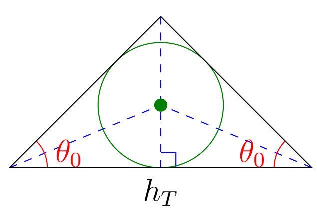
Solution:

Let T be a triangle and (1.1) be satisfied. Let θ_T be the smallest angle of T, A be the vertex of T corresponding to θ_T, and B, C the other two vertices. Define $b = |AC|$, and v as the distance of C from the line AB. Then,



$$\theta_T \leq \frac{\pi}{3} \quad \text{and} \quad \sin\theta_T = \frac{v}{b} \geq \frac{\varrho_T}{h_T} \geq \frac{1}{\sigma} \qquad \Longrightarrow \qquad \theta_T \geq \arcsin\frac{1}{\sigma} =: \theta_0.$$

Therefore, (1.1) $\implies$ all angles bounded from below.



Now assume all angles are bounded from below by $\theta_0 > 0$ and define T' as on the left. Then, for any $T \in \mathcal{T}_h$, T' is contained within T. As the centre of the inscribed circle of T' is at the intersection of the bisectors of the angles, its diameter is given by

$$h_T \tan\frac{\theta_0}{2} \leq \varrho_T.$$

Therefore,

$$\frac{h_T}{\varrho_T} \leq \frac{1}{\tan\frac{\theta_0}{2}} =: \sigma \qquad \forall T \in \mathcal{T}_h.$$

2. Consider a triangulation $\mathcal{T}_h$ of $\Omega \subset \mathbb{R}^n$ consisting of simplices T with diameter h_T, define by ϱ_T the diameter of the largest inscribed ball in T, and assume that (1.1) holds.

   (a) (1 point) Show that $|T|$, for any $T \in \mathcal{T}_h$, satisfies the condition

$$C_1 h_T^n \leq |T| \leq C_2 h_T^n,$$

   where C_1 is a positive constant dependent only on σ and n, and C_2 is a positive constant dependent only on n.

Solution:

Let Ω_n be the volume of the unit ball in $\mathbb{R}^n$; then, the volume of a ball with diameter d is $\Omega_n(d/2)^n$. As the volume of a simplex is larger than the volume of its inscribed ball

$$|T| \geq \Omega_n \left(\frac{\varrho_T}{2}\right)^n \geq \underbrace{\frac{\Omega_n}{(2\sigma)^n}}_{C_1} h_T^n. \tag{2.2}$$

Furthermore, as T is contained within any ball with radius h_T and centre in T,

$$|T| \leq \underbrace{\Omega_n}_{C_2} h_T^n \tag{2.3}$$

   (b) (1 point) Show, for $n = 3$, that any face F of $\mathcal{T}_h$ satisfies the condition.

$$\frac{h_F}{\varrho_F} \leq \sigma.$$

Solution:

Consider any simplex $T \in \mathcal{T}_h$ and let F be a face of T. Let $B \subset T$ be ball with centre c contained within T; therefore, its diameter is less or equal to ϱ_T. Let p be a plane parallel to F passing through c. Then, $p \cap T$ is a similar (smaller) triangle to F, and the circle $p \cap B$ has the same radius as B; therefore, $\varrho_F \geq \varrho_T$. As $h_F \leq h_T$ then

$$\frac{h_F}{\varrho_F} \leq \frac{h_T}{\varrho_T} \leq \sigma.$$

   (c) (1 point) Show that $h_T \leq \sigma h_{\widetilde{T}}$, for $n = 2, 3$, for any pair of elements $T, \widetilde{T} \in \mathcal{T}_h$ sharing an edge.

Solution:

Let ℓ be the length of the edge shared by T and $\widetilde{T}$. Then,

$$h_T \leq \varrho_T \sigma \leq \ell\sigma \leq h_{\widetilde{T}}\sigma.$$

3. (2 points) Let $\mathcal{T}_h$ be a triangulation consisting of n-simplices T in $\mathbb{R}^n$ satisfying (1.1) and the assumptions ($\mathcal{T}_h$1)–($\mathcal{T}_h$5). Prove that the number of elements of $\mathcal{T}_h$ sharing a vertex is bounded by a constant depending on σ and n.

Solution:

Let a be a vertex of $\mathcal{T}_h$ and let $\mathcal{M}_a = \{T \in \mathcal{T}_h : a \in T\}$ be the set of simplices sharing the vertex a. Consider any $T \in \mathcal{M}_a$ with vertices $a_1, \ldots, a_{n+1}$, and let $\widetilde{T}$ be a simplex with vertices $\widetilde{a}_1, \ldots, \widetilde{a}_{n+1}$ satisfying $(a_i - a) = h_T(\widetilde{a}_i - a)$, $i = 1, \ldots, n+1$. Then, $\widetilde{T} = \phi(T)$, where $\phi(x) = {}^{(x-a)}/_{h_T} + a$, and $h_{\widetilde{T}} = 1$. For a ball $B \subset T$, then $\widetilde{B} := \phi(B)$ is a ball in $\widetilde{T}$ and $\operatorname{diam}(B) = h_T \operatorname{diam} \widetilde{B}$; therefore,

$$\sigma \geq \frac{h_T}{\varrho_T} = \frac{h_T}{h_T \varrho_{\widetilde{T}}} = \frac{h_{\widetilde{T}}}{\varrho_{\widetilde{T}}}.$$

Denoting by Ω_n the volume of the unit ball in $\mathbb{R}^n$, we have that

$$\Omega_n \geq \sum_{T \in \mathcal{M}_a} |\widetilde{T}| \geq \sum_{T \in \mathcal{M}_a} \left(\frac{\varrho_{\widetilde{T}}}{2}\right)^n \Omega_n \geq \sum_{T \in \mathcal{M}_a} \left(\frac{1}{2\sigma}\right)^n \Omega_n = \left(\frac{1}{2\sigma}\right)^n \Omega_n \operatorname{card} \mathcal{M}_a;$$

therefore, $\operatorname{card} \mathcal{M}_a \leq (2\sigma)^n$.