06.11.2023 — Homework 2

Finite Element Methods 1

Due date: 20th November 2023

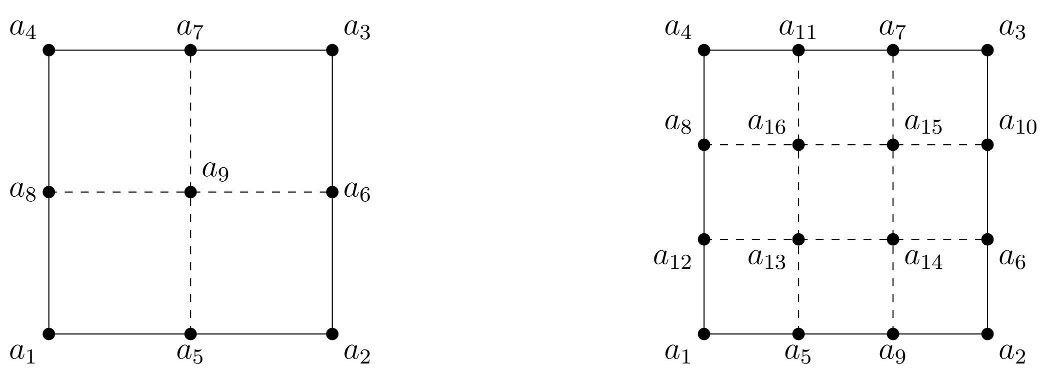
Submit a PDF/scan of the answers to the following questions before the deadline via the *Study Group Roster* (*Záznamník učitele*) in SIS, or hand-in directly at the practical class on the 20th November 2023.

1. (2 points) Consider finite elements (T, P_T, Σ_T), where

$$\begin{aligned} &T \text{ is a rectangle,} \\ &P_T = Q_3(T), \\ &\Sigma_T = \{p(z) : z \in M_3(T)\}. \end{aligned}$$

For $T = [0,1]^2$, and the points from the principal lattice $M_3(T)$ numbered as per Figure 1b, write basis functions of the finite element (T, P_T, Σ_T). It is sufficient to derive functions for only four basis functions, as the remaining twelve can be obtained by circular permutations of the indices. Let $\mathcal{T}_h$ be a triangulation of a bounded domain $\Omega \subset \mathbb{R}^2$ consisting of rectangles and assign the above finite element to each $T \in \mathcal{T}_h$. Write the definition of the corresponding finite element space X_h and verify that $X_h \subset C(\overline{\Omega})$.

Solution:



(a) $M_2(T)$ (b) $M_3(T)$

Figure 1: Principal lattices for rectangles

Using the notation and coordinate systems introduced in the practical class ($x_3 = 1 - x_1$ and $x_4 = 1 - x_2$), we define the basis functions on $T = [0, 1]^2$ as

$$p_1 = \frac{1}{4} x_3(3x_3 - 1)(3x_3 - 2)x_4(3x_4 - 1)(3x_4 - 2), \qquad p_2, p_3, p_4 \text{ by circ. perm.,}$$
$$p_5 = -\frac{9}{4} x_3(3x_3 - 1)(x_3 - 1)x_4(3x_4 - 1)(3x_4 - 2), \qquad p_6, p_7, p_8 \text{ by circ. perm.,}$$
$$p_9 = \frac{9}{4} x_3(3x_3 - 2)(x_3 - 1)x_4(3x_4 - 1)(3x_4 - 2), \qquad p_{10}, p_{11}, p_{12} \text{ by circ. perm.,}$$
$$p_{13} = \frac{81}{4} x_3(3x_3 - 1)(x_3 - 1)x_4(3x_4 - 1)(x_4 - 1), \qquad p_{14}, p_{15}, p_{16} \text{ by circ. perm..}$$

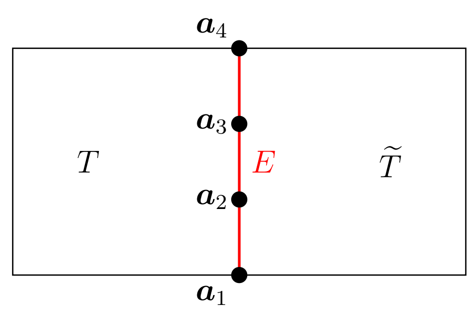
In order to define the finite element space we let

$$\bigcup_{T \in \mathcal{T}_h} M_3(T) = \{z_i\}_{i=1}^{N_h}, \qquad \text{and} \qquad \mathcal{T}_h^i = \{T \in \mathcal{T}_h : z_i \in T\}, \quad i = 1, \ldots, N_h.$$

Then, the finite element space is defined as

$$X_h = \{v_h \in L^2(\Omega) : v_h|_T \in Q_3(T)\ \forall T \in \mathcal{T}_h,$$
$$v_h|_T(z_i) = v_h(z_i)|_{\widetilde{T}}\ \forall T, \widetilde{T} \in \mathcal{T}_h^i, i = 1, \ldots, N_h\}.$$

In order to show continuity of the space we let $v_h \in X_h$ and consider *any* interior edge E of $\mathcal{T}_h$. Let $T, \widetilde{T} \in \mathcal{T}_h$ be the two elements of T_h, sharing the edge E, and denote by $a_1, \ldots, a_4$ the four points of $\{z_i\}_{i=1}^{N_h}$ lying on the edge E, see figure to the right. Then, $v_h|_T(a_i) = v_h|_{\widetilde{T}}(a_i)$, $i = 1, \ldots, 4$. Defining $p = (v_h|_T)|_E - (v_h|_{\widetilde{T}})|_E$, we have that $p \in P_3(E)$ and $p(a_i) = 0$, $i = 1, \ldots, 4$. Therefore, $p \equiv 0$ and, hence, v_h is continuous across E. As E was arbitrary we deduce that $v_h \in C(\overline{\Omega})$.



2. (2 points) Let the points $a_1, \ldots a_9$ be the points of the principal lattice $M_2(T)$, see Figure 1a, and define the space

$$Q_2'(T) = \left\{ p \in Q_2(T) : 4\,p(a_9) + \sum_{i=1}^{4} p(a_i) - 2 \sum_{i=5}^{8} p(a_i) = 0 \right\}.$$

Show that any polynomial $p \in Q_2'(T)$ is uniquely determined by the values at the points $a_1, \ldots, a_8$ and derive basis functions $p_1', \ldots, p_8'$ of $Q_2'(T)$ satisfying $p_i'(a_j) = \delta_{ij}$, $i, j = 1, \ldots, 8$. Prove that $P_2(T) \subset Q_2'(T)$.

Hint. We can proceed similarly as for the reduced Lagrange cubic n-simplex. It is sufficient to derive functions for only two basis functions, as the remaining six can be obtained by circular permutations of the indices.

Solution:

Let $p_1, \dots, p_9$ be the basis functions of $Q_2(T)$, with $p_i(a_j) = \delta_{ij}$, $i, j = 1, \dots, 9$. We can define $p'_i := p_i + \alpha_i p_9$, for $i = 1, \dots, 8$, which we require to be functions in $Q'_2(T)$; therefore, we have that

$$
\begin{aligned}
0 &= 4p'(a_9) + \sum_{i=1}^{4} p'(a_i) - 2\sum_{i=5}^{8} p'(a_i) = 4\alpha_i + 1, \qquad && \text{for } i = 1, \dots, 4,\\
0 &= 4p'(a_9) + \sum_{i=1}^{4} p'(a_i) - 2\sum_{i=5}^{8} p'(a_i) = 4\alpha_i - 2, \qquad && \text{for } i = 5, \dots, 8.
\end{aligned}
$$

Hence, selecting $\alpha_i = -1/4$, for $i = 1, \dots, 4$ and $\alpha_i = 1/2$, for $i = 5, \dots, 8$, we have that $p'_1, \dots, p'_8 \in Q'_2(T)$. Furthermore $p'_1, \dots, p'_8, p_9$ form a basis of $Q_2(T)$ (nine linearly independent functions in $Q_2(T)$), and since $p_9 \notin Q'_2$ then $p'_1, \dots, p'_8$ form a basis of $Q'_2(T)$. As $p'_i(a_j) = \delta_{ij}$, $i, j = 1, \dots, 8$ then $p \in Q'_2(T)$ is uniquely determined by its values at the points $a_1, \dots, a_8$. Using the definition of p'_i, p_i and the value of α_i we have that

$$
\begin{aligned}
p'_1 &= x_3(2x_3 - 1)x_4(2x_4 - 1) - 4x_1x_2x_3x_4 = x_3x_4(2x_3 + 2x_4 - 3),\\
p'_5 &= -4x_3(x_3 - 1)x_4(2x_4 - 1) + 8x_1x_2x_3x_4 = -4x_3x_4(x_3 - 1),
\end{aligned}
$$

where x_1, x_2, x_3, x_4 are defined as in the practical class. The other basis functions are defined by circular permutations of the indices.

To show $P_2(T) \subset Q'_2(T)$ we let $p \in P_2(T)$ and show that $p \in Q'_2(T)$. Denoting by $A = \nabla^2 p$ the Hessian of p, and applying the (exact) Taylor's formula around a_9 we have that

$$
p(a_i) = p(a_9) + \nabla p(a_9) \cdot (a_i - a_9) + \frac{1}{2}(a_i - a_9) \cdot A(a_i - a_9), \qquad i = 1, \cdots, 8.
$$

Noting that as a_9 is the barycentre of the rectangle

$$
a_9 = \frac{a_1 + a_2 + a_3 + a_4}{4} = \frac{a_5 + a_6 + a_7 + a_8}{4} \quad \Longrightarrow \quad \sum_{i=1}^{4}(a_i - a_9) = \sum_{i=5}^{8}(a_i - a_9) = 0;
$$

then,

$$
\begin{aligned}
\sum_{i=1}^{4} p(a_i) &= 4p(a_9) + \frac{1}{2}\sum_{i=1}^{4}(a_i - a_9) \cdot A(a_i - a_9),\\
\sum_{i=5}^{8} p(a_i) &= 4p(a_9) + \frac{1}{2}\sum_{i=5}^{8}(a_i - a_9) \cdot A(a_i - a_9).
\end{aligned}
$$

As $a_5, \ldots, a_8$ are the midpoints of the edges we have that $a_i = \frac{1}{2}(a_{i-4} + a_{i-3})$, for $i = 5, 6, 7$ and $a_8 = \frac{1}{2}(a_4 + a_1)$; therefore,

$$\sum_{i=5}^{8}(a_i - a_9) \cdot A(a_i - a_9) = \frac{1}{2}\sum_{i=1}^{4}(a_i - a_9) \cdot A(a_i - a_9) + \frac{1}{2}\sum_{i=1}^{3}(a_i - a_9) \cdot A(a_{i+1} - a_9) + \frac{1}{2}(a_4 - a_9) \cdot A(a_1 - a_9).$$

Considering only the last two terms we have that

$$\begin{aligned}(a_1 - a_9) \cdot A(a_2 - a_9) + (a_2 - a_9) \cdot A(a_3 - a_9) \\ + (a_3 - a_9) \cdot A(a_4 - a_9) + (a_4 - a_9) \cdot A(a_1 - a_9) \\ = (a_2 - a_9) \cdot A(a_1 + a_3 - 2a_9) + (a_4 - a_9) \cdot A(a_1 + a_3 - 2a_9) = 0,\end{aligned}$$

as a_9 is the midpoint of a_1 and a_3. Combining these results we have that

$$\sum_{i=1}^{4} p(a_i) - 2\sum_{i=5}^{8} p(a_i) = -4p(a_9),$$

and, hence, $p \in Q_2'(T)$.

3. (2 points) Let T be a pentahedral prism, see Figure 2, with vertices $a_1, \ldots, a_6$. The triangular faces are orthogonal to the x_3 axis, and the quadrilateral faces are parallel to the x_3 axis. Let

$$\begin{aligned}P_T = \{p(x_1, x_2, x_3) = \gamma_1 + \gamma_2 x_1 + \gamma_3 x_2 + \gamma_4 x_3 \\ + \gamma_5 x_1 x_3 + \gamma_6 x_2 x_3 \\ : \gamma_1, \ldots, \gamma_6 \in \mathbb{R}\}.\end{aligned}$$

Show that any function $p \in P_T$ is uniquely determined by its values at the vertices $a_1, \ldots, a_6$ and that, for any $p \in P_T$ and face $F \subset \partial T$, the restriction $p|_F$ is uniquely determined by its values at the vertices of the face F.

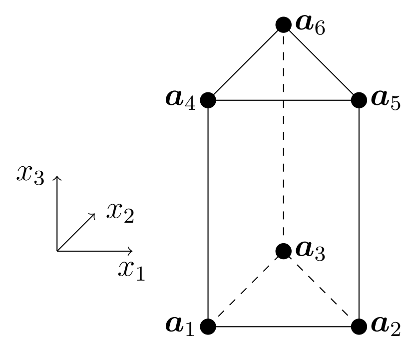
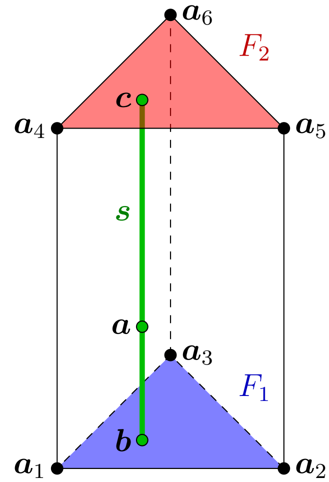


Figure 2: Pentahedral prism

Solution:

Let F_1 be the triangle defined by the vertices a_1, a_2, a_3, F_2 be the triangle defined by the vertices a_4, a_5, a_6, and F_3, F_4, F_5 be the three quadrilateral faces. For any $p \in P_T$ we have that $p|_{F_1} \in P_1(F_1)$ and hence $p|_{F_1}$ is uniquely determined by $p(a_1)$, $p(a_2)$, and $p(a_3)$. Similarly, $p|_{F_2}$ is uniquely determined by $p(a_4)$, $p(a_5)$, and $p(a_6)$. We now consider an arbitrary point $a \in T$ and draw a line $\boldsymbol{s}$ through a in the x_3 direction, with endpoints $b \in F_1$ and $c \in F_2$. Since $p|_{\boldsymbol{s}} \in P_1(\boldsymbol{s})$, then $p|_{\boldsymbol{s}}$, and $p(a)$, are uniquely determined by the values $p(b)$ and $p(c)$, which are in turn uniquely determined by the values at $a_1, \ldots, a_6$ as shown above; therefore, any function $p \in P_T$ is uniquely determined by its values at the vertices.



We have also shown that $p|_{F_1}$ and $p|_{F_2}$ are uniquely determined by the values of p at the vertices of F_1 and F_2, respectively; therefore, we only need to show that p restricted to a quadrilateral face is uniquely determined by its values at the vertices of the face. We consider the face F with vertices a_1, a_2, a_5, and a_4, noting that the other two faces follow analogously. Define a coordinate $\widetilde{x}_1$ in the direction of the edge E with endpoints a_1 and a_2; then, along E we have that

$$(x_1, x_2) = (\alpha_1 \widetilde{x}_1 + \beta_1, \alpha_2 \widetilde{x}_1 + \beta_2),$$

where $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}$. Then, we have that

$$\begin{aligned} p|_F &= \gamma_1 + \gamma_2(\alpha_1 \widetilde{x}_1 + \beta_1) + \gamma_3(\alpha_2 \widetilde{x}_1 + \beta_2) + \gamma_4 x_3 + \gamma_5(\alpha_1 \widetilde{x}_1 + \beta_1)x_3 + \gamma_6(\alpha_2 \widetilde{x}_1 + \beta_2)x_3 \\ &= \delta_1 + \delta_2 \widetilde{x}_1 + \delta_3 x_3 + \delta_4 \widetilde{x}_1 x_3, \end{aligned}$$

where

$$\begin{aligned} \delta_1 &= \gamma_1 + \gamma_2 \beta_1 + \gamma_3 \beta_2, \\ \delta_2 &= \gamma_2 \alpha_1 + \gamma_3 \alpha_2, \\ \delta_3 &= \gamma_4 + \gamma_5 \beta_1 + \gamma_6 \beta_2, \\ \delta_4 &= \gamma_5 \alpha_1 + \gamma_6 \alpha_2. \end{aligned}$$

As $\widetilde{x}_1$ and x_3 define a coordinate system for F we have that $p|_F \in Q_1(F)$ and, hence, is uniquely determined by its values at the vertices of F.