# Nonlinear Differential Equations

## Practical 9: Brouwer Fixed Point

1. Prove Theorem 3.6 from the lecture:

   **Theorem 3.6.** *Let $X$ be a finite-dimensional normed linear space $K \subset X$ a closed, convex, and bounded subset, and $f : K \to K$ a continuous mapping. Then, there exists a fixed point of $f$ in $K$; i.e., $\exists \overline{x} \in K$ such that*
   $$\overline{x} = f(\overline{x}).$$

   **Hint.** *Define*
   $$x = \sum_{i=1}^{n} \alpha_i x_i,$$
   *where $x_1, \ldots, x_n$ form a basis for $X$ ($\dim X = n$), and $\boldsymbol{\alpha} = \{\alpha_i\}_i^n \in \mathbb{R}^n$. Then, define a linear, continuous operator $T : X \to \mathbb{R}^n$ as $T(x) = \boldsymbol{\alpha}$ (which has a continuous inverse $T^{-1}$). Defining $K_1 = T(K)$ and $g(\boldsymbol{\alpha}) = T \circ f \circ T^{-1}\boldsymbol{\alpha}$, show that $T$ is a homeomorphism and $g$ has a fixed point, and hence, that $f$ has a fixed point.*

2. Let the conditions of Theorem 2.11/Corollary 3.8 be met; i.e., $A : X \to X'$ monotone, coercive, and hemicontinuous on a real *separable* reflexive Banach space $X$.

   (a) Show that if $A$ is strictly monotone that the inverse $A^{-1}$ exists and is strictly monotone, demicontinuous, and bounded.

   **Hint.** *For demicontinuous, let $v_n = A^{-1} f_n$, $f_n \to f$. Show sequence $\{v_n\}$ is bounded; hence, there exists a subsequence $v_{n'} \rightharpoonup v$, and show $v = A^{-1} f$. Then, show holds for whole sequence (see Proposition 1.8).*

   (b) If $A$ is uniformly monotone, show that $A^{-1}$ is continuous.

   (c) If $A$ is strongly monotone, show that $A^{-1}$ is Lipschitz continuous.

3. Let $A : X \to X'$ be a bounded operator on a real, separable, reflexive, and infinite-dimensional Banach space $X$ and $f \in X'$. Let $\{v_1, v_2, \ldots\}$ be the basis of $X$ and there exists a $R > 0$ and $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$
   $$\langle Au_n - f, v_k \rangle = 0, \qquad u_n \in X_n, \qquad k = 1, \ldots, n, \tag{3.1}$$
   where $X_n = \operatorname{span}\{v_1, \ldots, v_n\}$ has a solution $u_n$ with $\|u_n\| \leq R$.

   (a) If $A$ satisfies (M), show that there exists a subsequence $\{u_{n'}\}$ of $\{u_n\}$ with $u_{n'} \rightharpoonup u$ such that $u \in X$ is a solution of $Au = f$.

   **Hint.** *Consider the limit as $n \to \infty$ of* (3.1) *and show the left hand side of* (M) *is satisfied.*

   (b) If $A$ satisfies $(\mathrm{S})_0$ and is demicontinuous, show that there exists a subsequence $\{u_{n'}\}$ of $\{u_n\}$ with $u_{n'} \to u$ such that $u \in X$ is a solution of $Au = f$.

   **Hint.** *Show left hand side of* $(\mathrm{S})_0$ *is satisfied.*