Nonlinear Differential Equations

Practical 8: Galerkin Approximation I

1. Let $A : X \to Y$ be *linear* and continuous on Banach spaces X and Y. Show that if there exists a sequence $\{u_n\}$ such that $u_n \rightharpoonup u$ then $Au_n \rightharpoonup Au$.

   Hint. *Use the fact that there exists a dual operator $A^d \in \mathcal{L}(Y', X')$ to A; see Section 1.3 of notes.*

   Solution: As A is linear there exists $A^d \in \mathcal{L}(Y', X')$ such that for $v \in Y'$

   $$\langle v, Ax \rangle_{Y' \times Y} = \langle A^d v, x \rangle_{X' \times X} \qquad \forall x \in X.$$

   Therefore, from $u_n \rightharpoonup u$, we have that for all $v \in Y'$

   $$\langle v, Au_n - Au \rangle = \langle v, A(u_n - u) \rangle = \langle A^d v, u_n - u \rangle \to 0;$$

   hence, $Au_n \rightharpoonup Au$.

2. Let $X_n \subset X$ be a finite dimensional subspace of a separable Banach space X, $A : X \to X'$ and $A_n : X_n \to X_n'$, where $A_n = P_n^d A P_n$ given the *linear* and *continuous* projection $P_n : X \to X_n$. Show:

   (a) A continuous $\implies$ A_n continuous

   Solution: Let $u_m \to u$ in X_n; then, $P_n u_n \to P_n u$ as P_n is continuous. From A continuous we get that $AP_n u_m \to AP_n u$, and finally as P_n^d is also continuous $A_n u_m = P_n^d A P_n u_n \to P_n^d A P_n u = A_n u$.

   (b) A weakly continuous $\implies$ A_n weakly continuous

   Solution: Let $u_m \rightharpoonup u$ in X_n; then, as P_n is linear and continuous, by Question 1 $P_n u_m \rightharpoonup P_n u$. From A weakly continuous we get that $AP_n u_m \rightharpoonup AP_n u$, and finally as P_n^d is also linear and continuous $A_n u_m = P_n^d A P_n u_m \rightharpoonup P_n^d A P_n u = A_n u$ by Question 1.

   (c) A strongly continuous $\implies$ A_n strongly continuous

   Solution: Let $u_m \rightharpoonup u$ in X_n; then, as P_n is linear and continuous, by Question 1 $P_n u_m \rightharpoonup P_n u$. From A strongly continuous we get that $AP_n u_m \to AP_n u$, and finally as P_n^d is also linear and continuous $A_n u_m = P_n^d A P_n u_m \to P_n^d A P_n u = A_n u$.

   (d) A demicontinuous $\implies$ A_n demicontinuous

   Solution: Let $u_m \to u$ in X_n; then, $P_n u_n \to P_n u$ as P_n is continuous. From A demicontinuous we get that $AP_n u_m \rightharpoonup AP_n u$, and finally as P_n^d is linear and continuous $A_n u_m = P_n^d A P_n u_m \rightharpoonup P_n^d A P_n u = A_n u$ by Question 1.

(e) A hemicontinuous $\implies A_n$ hemicontinuous

Solution: Let $t_m \to 0$ then, for all $v \in X_n$,

$$\begin{aligned}\langle A_n(u+t_m v), v\rangle &= \langle P_n^d A P_n(u+t_m v), v\rangle \\ &= \langle A(P_n u + t_m P_n v), P_n v\rangle \to \langle AP_n u, P_n v\rangle = \langle A_n u, v\rangle.\end{aligned}$$

(f) A Lipschitz continuous $\implies A_n$ Lipschitz continuous

Solution: For all $u, v, w \in X_n$

$$\begin{aligned}\langle A_n u - A_n v, w\rangle &= \langle P_n^d (AP_n u - AP_n v), w\rangle \\ &= \langle AP_n u - AP_n v, P_n w\rangle \\ &\leq \|AP_n u - AP_n v\|_{X'} \|P_n w\|_X \\ &\leq L\|P_n(u-v)\|_X \|P_n w\|_X \\ &= L\|u-v\|_{X_n}\|w\|_{X_n}.\end{aligned}$$

Then,

$$\|A_n u - A_n v\|_{X_n'} = \sup_{w\in X_n} \frac{|\langle A_n u - A_n v, w\rangle|}{\|w\|_{X_n}} \leq L\|u-v\|_{X_n}.$$

(g) A monotone $\implies A_n$ monotone

Solution: For all $u, v \in X_n$,

$$\langle A_n u - A_n v, u-v\rangle = \langle P_n^d(AP_n u - AP_n v), u - v\rangle = \langle AP_n u + AP_n v, P_n u - P_n v\rangle \geq 0.$$

(h) A strongly monotone $\implies A_n$ strongly monotone

Solution: For all $u, v \in X_n$,

$$\begin{aligned}\langle A_n u - A_n v, u - v\rangle &= \langle P_n^d(AP_n u - AP_n v), u - v\rangle \\ &= \langle AP_n u + AP_n v, P_n u - P_n v\rangle \\ &\geq M\|P_n(u-v)\|_X^2 \\ &= M\|u-v\|_{X_n}.\end{aligned}$$

3. Consider the boundary value problem, on $\Omega = (0,1) \subset \mathbb{R}$,

$$\begin{aligned}-\frac{\mathrm{d}}{\mathrm{d}x}\left(\mu(x, |u'(x)|)u'(x)\right) = f \qquad &\text{in } (0,1), \\ u(0) = u(1) = 0, &\end{aligned}$$

where $\mu(x,t) = 1 + e^{-t}$. Note that, there exists $\alpha_1 \geq \alpha_2 > 0$ such that for $t \geq s \geq 0$ and $x \in [0,1]$

$$\alpha_2(t-s) \leq \mu(x,t)t - \mu(x,s)s \leq \alpha_1(t-s).$$

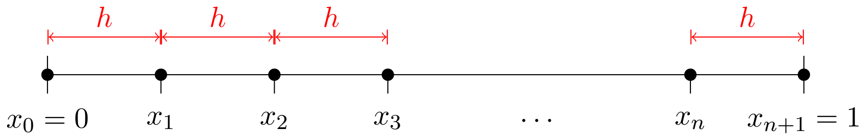

Figure 1: Finite element mesh for interval $(0, 1)$ with $n + 1$ nodes

Let $X = H_0^1(0, 1)$ with norm $\|\cdot\|_X = |\cdot|_{1,2}$, and inner product

$$(u, v)_X := \int_0^1 u'v' \,\mathrm{d}x.$$

Then, as this is similar to Examples 2.1 and 3.1 we have a unique weak solution $u \in H_0^1(0, 1)$ to the weak formulation

$$a(u, v) := \int_0^1 \mu(x, |u'|)u'v' \,\mathrm{d}x = \int_0^1 fv \,\mathrm{d}\boldsymbol{x} =: \langle F, v\rangle \qquad \text{for all } v \in H_0^1(0, 1),$$

and that there exists a $A \in H^{-1}(0, 1)$ such that $\langle Au, v\rangle = a(u, v)$. Dividing the interval $(0, 1)$ into $n + 1$ intervals of equal length $h = {}^1/_{(n+1)}$ with nodes $x_i = ih$, $i = 0, \ldots, n + 1$, see Figure 1, we can define a finite dimensional subspace

$$X_n := \{v \in H_0^1(0, 1) : v|_{(x_i,x_{i+1})} \in P_1(x_i, x_{i+1}), i = 0, \ldots, n, v(0) = 0, v(1) = 0\} \subset H_0^1(0, 1),$$

where $P_1(x_i, x_{i+1})$ is the space of polynomials of degree one on the interval (x_i, x_{i+1}). We can define the basis functions $\{\phi_i\}_{i=1}^n$ of X_n as

$$\phi_i(x) = \begin{cases} 1 + {}^{(x-x_i)}/_h & \text{if } x_{i-1} \le x \le x_i, \\ 1 - {}^{(x-x_i)}/_h & \text{if } x_i \le x \le x_{i+1}, \\ 0 & \text{otherwise.} \end{cases}$$

Define the iterative Galerkin finite element approximation, similar to Example 3.1, to find a sequence $\{u_n^{(m)}\}_{m\ge0} \subset X_n$ which converges to the approximation $u_n \in X_n$ for a starting $u_n^{(0)}$. Furthermore, state the iteration as an algebraic linear system for $n = 10$.

Solution: Following Example 3.1, we can define the iterative Galerkin FEM: Given an initial guess $u_n^{(0)} \in X_n$ we iterate for $m = 0, 1, 2, \ldots$ and find $u_n^{(m+1)}$ such that

$$(u_n^{(m+1)}, v)_X = (u_n^{(m)}, v)_X - \frac{\alpha_2}{\alpha_1}\langle Au_n^{(m)} - F, v\rangle \qquad \forall v \in X_n.$$

Using the definition of the basis functions, and defining

$$u_n^{(m)} = \sum_{j=1}^n \beta_n^{(m)}\phi_j, \qquad \text{for } m = 0, 1, 2, \ldots$$

where $\boldsymbol{\beta}^{(m)} = (\beta_1^{(m)}, \ldots, \beta_n^{(m)}) \in \mathbb{R}^n$, $m = 0, 1, 2, \ldots$, we can define as a linear system: find $\boldsymbol{\beta}^{(m+1)} \in \mathbb{R}^n$ such that

$$\sum_{j=1}^n \beta_n^{(m+1)}(\phi_j, \phi_i)_X = \sum_{j=1}^n \beta_n^{(m)}(\phi_j, \phi_i)_X - \frac{\alpha_2}{\alpha_1}\left\langle A\left(\sum_{j=1}^n \beta_j^{(m)}\phi_j\right) - F, \phi_i\right\rangle,$$

for $j = 1, \dots, n$. Alternatively, as

$$\mathbb{M}\boldsymbol{\beta}^{(m+1)} = \mathbb{M}\boldsymbol{\beta}^{(m)} - \frac{\alpha_2}{\alpha_1}\boldsymbol{F}(\boldsymbol{\beta}^{(m)})$$

where $\mathbb{M} \in \mathbb{R}^{n\times n}$ and $\boldsymbol{F} : \mathbb{R}^n \to \mathbb{R}^n$ are defined for $i, j = 1, \dots, n$,

$$\begin{aligned}
M_{i,j} &= (\phi_j, \phi_i)_X = \int_0^1 \phi_j' \phi_i' \,\mathrm{d}x \\
\boldsymbol{F}_i(\boldsymbol{\beta}^{(m)}) &= \left\langle A\left(\sum_{j=1}^n \beta_j^{(m)}\phi_j\right) - F, \phi_i \right\rangle \\
&= \int_0^1 \left(1 + \exp\left(-\left|\sum_{j=1}^n \beta_j^{(m)}\phi_j\right|\right)\right) \sum_{j=1}^n \beta_j^{(m)} \phi_j' \phi_i' \,\mathrm{d}x - \int_0^1 f\phi_i \,\mathrm{d}x.
\end{aligned}$$

Noting that

$$\phi_i('x) = \begin{cases} 1/h & \text{if } x_{i-1} \le x \le x_i, \\ -1/h & \text{if } x_i \le x \le x_{i+1}, \\ 0 & \text{otherwise.} \end{cases}$$

Expanding these definitions, we have that

$$\begin{aligned}
M_{i,i} &= \int_{x_{i-1}}^{x_i} \phi_j'\phi_i' \,\mathrm{d}x + \int_{x_i}^{x_{i+1}} \phi_j'\phi_i' \,\mathrm{d}x = \frac{2}{h} \\
M_{i,i+1} &= \int_{x_i}^{x_{i+1}} \phi_{i+1}'\phi_i' \,\mathrm{d}x = -\frac{1}{h} \\
M_{i,i-1} &= \int_{x_{i-1}}^{x_i} \phi_{i-1}'\phi_i' \,\mathrm{d}x = -\frac{1}{h} \\
M_{i,j} &= 0, && \text{if } j \notin \{i-1, i, i+1\}.
\end{aligned}$$

Therefore, for $n = 10$, $h = \frac{1}{11}$, and

$$\mathbb{M} = 11 \begin{pmatrix}
2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2
\end{pmatrix}.$$

We note that $\boldsymbol{F}_i(\boldsymbol{\beta}^{(m)})$ would probably need computation via numerical quadrature (or similar) at each iteration due to the nonlinearity.