Nonlinear Differential Equations

Practical 8: Galerkin Approximation I

1. Let $A : X \to Y$ be *linear* and continuous on Banach spaces X and Y. Show that if there exists a sequence $\{u_n\}$ such that $u_n \rightharpoonup u$ then $Au_n \rightharpoonup Au$.

   Hint. *Use the fact that there exists a dual operator $A^d \in \mathcal{L}(Y', X')$ to A; see Section 1.3 of notes.*

2. Let $X_n \subset X$ be a finite dimensional subspace of a separable Banach space X, $A : X \to X'$ and $A_n : X_n \to X_n'$, where $A_n = P_n^d A P_n$ given the *linear* and *continuous* projection $P_n : X \to X_n$. Show:
   (a) A continuous $\implies$ A_n continuous
   (b) A weakly continuous $\implies$ A_n weakly continuous
   (c) A strongly continuous $\implies$ A_n strongly continuous
   (d) A demicontinuous $\implies$ A_n demicontinuous
   (e) A hemicontinuous $\implies$ A_n hemicontinuous
   (f) A Lipschitz continuous $\implies$ A_n Lipschitz continuous
   (g) A monotone $\implies$ A_n monotone
   (h) A strongly monotone $\implies$ A_n strongly monotone

3. Consider the boundary value problem, on $\Omega = (0,1) \subset \mathbb{R}$,

$$-\frac{\mathrm{d}}{\mathrm{d}x}\left(\mu(x, |u'(x)|)u'(x)\right) = f \qquad \text{in } (0,1),$$
$$u(0) = u(1) = 0,$$

where $\mu(x,t) = 1 + e^{-t}$. Note that, there exists $\alpha_1 \geq \alpha_2 > 0$ such that for $t \geq s \geq 0$ and $x \in [0,1]$

$$\alpha_2(t-s) \leq \mu(x,t)t - \mu(x,s)s \leq \alpha_1(t-s).$$

Let $X = H_0^1(0,1)$ with norm $\|\cdot\|_X = |\cdot|_{1,2}$, and inner product

$$(u,v)_X := \int_0^1 u'v' \,\mathrm{d}x.$$

Then, as this is similar to Examples 2.1 and 3.1 we have a unique weak solution $u \in H_0^1(0,1)$ to the weak formulation

$$a(u,v) := \int_0^1 \mu(x, |u'|)u'v' \,\mathrm{d}x = \int_0^1 fv \,\mathrm{d}\boldsymbol{x} =: \langle F, v\rangle \qquad \text{for all } v \in H_0^1(0,1),$$

and that there exists a $A \in H^{-1}(0,1)$ such that $\langle Au, v\rangle = a(u,v)$. Dividing the interval $(0,1)$ into $n+1$ intervals of equal length $h = 1/(n+1)$ with nodes $x_i = ih$, $i = 0, \ldots, n+1$, see Figure 1, we can define a finite dimensional subspace

$$X_n := \{v \in H_0^1(0,1) : v|_{(x_i, x_{i+1})} \in P_1(x_i, x_{i+1}), i = 0, \ldots, n, v(0) = 0, v(1) = 0\} \subset H_0^1(0,1),$$

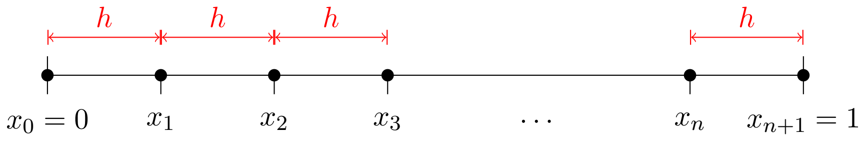

Figure 1: Finite element mesh for interval $(0, 1)$ with $n + 1$ nodes

where $P_1(x_i, x_{i+1})$ is the space of polynomials of degree one on the interval (x_i, x_{i+1}). We can define the basis functions $\{\phi_i\}_{i=1}^n$ of X_n as

$$\phi_i(x) = \begin{cases} 1 + {}^{(x-x_i)}\!/_{h} & \text{if } x_{i-1} \leq x \leq x_i, \\ 1 - {}^{(x-x_i)}\!/_{h} & \text{if } x_i \leq x \leq x_{i+1}, \\ 0 & \text{otherwise.} \end{cases}$$

Define the iterative Galerkin finite element approximation, similar to Example 3.1, to find a sequence $\{u_n^{(m)}\}_{m\geq 0} \subset X_n$ which converges to the approximation $u_n \in X_n$ for a starting $u_n^{(0)}$. Furthermore, state the iteration as a algebraic linear system for $n = 10$.