Nonlinear Differential Equations

Practical Exercises 8

Due: 17th April 2024

1. Let

$$(\mathcal{A}u)(x) = \sum_{|\alpha| \le k} D^\alpha a_\alpha(x, \delta_k u(x)),$$

$p \in (1, \infty)$, and $a_\alpha \in \mathrm{CAR}^*(p)$ for $|\alpha| \le k$. Let V be such that

$$W_0^{k,p}(\Omega) \subset V \subset W^{k,p}(\Omega),$$

Q a Banach space of functions in Ω, with norm $\|\cdot\|_Q$, where $C^\infty(\Omega)$ is dense in Q and V is continuously embedded in Q ($V \hookrightarrow Q$). Finally,

(a) function $\varphi \in W^{k,p}(\Omega)$,

(b) functional $g \in V^*$ such that for all $v \in W_0^{k,p}(\Omega)$, $\langle g, v \rangle_V = 0$,

(c) functional $f \in Q^*$.

Define $A : W^{k,p}(\Omega) \to (W^{k,p}(\Omega))^*$ such that for all $u, v \in W^{k,p}(\Omega)$

$$\langle Au, v \rangle = \sum_{|\alpha| \le k} \int_\Omega a_\alpha(x, \delta_k u(x)) D^\alpha v(x) \, \mathrm{d}\boldsymbol{x}.$$

Define the operator T on V such that, for $u \in V$, Tu is an element from V^* defined by

$$\langle Tu, v \rangle = \langle A(u + \varphi), v \rangle - \langle f, v \rangle_Q - \langle g, v \rangle_V \qquad \text{for all } v \in V.$$

Prove that

(a) T is bounded and demicontinuous (Lemma 3.14);
Hint. Use the continuity of the Nemyckii operator corresponding to a_α (see Theorem 3.11).

(b) T is monotone if, for all $\xi, \eta \in \mathbb{R}^\kappa$ and almost all $x \in \Omega$,

$$\sum_{|\alpha| \le k} \left(a_\alpha(x, \xi) - a_\alpha(x, \eta)\right)(\xi_\alpha - \eta_\alpha) \ge 0, \tag{1}$$

(Lemma 3.15)

(c) T is strictly monotone if equality only holds in (1) for $\eta = \xi$ (Corollary 3.16).

2. Consider the following Dirichlet problem:

$$\begin{aligned} -\nabla \cdot (\mu(\boldsymbol{x}, \nabla u) \nabla u) + b(\boldsymbol{x}, u) &= h(\boldsymbol{x}) && \text{in } \Omega \subset \mathbb{R}^2 \\ u &= 0 && \text{on } \partial\Omega \end{aligned}$$

where Ω has Lipschitz boundary, $u : \Omega \to \mathbb{R}$ is the unknown function, $h \in C(\overline{\Omega})$, and

$$\nabla \phi = \begin{pmatrix} \frac{\partial \phi}{\partial x_1} \\ \frac{\partial \phi}{\partial x_1} \end{pmatrix} \qquad \text{for scalar-valued function } \varphi : \Omega \to \mathbb{R},$$

$$\nabla \cdot \boldsymbol{\sigma} = \frac{\partial \sigma_1}{\partial x_1} + \frac{\partial \sigma_2}{\partial x_2} \qquad \text{for vector-valued function } \boldsymbol{\sigma} = (\sigma_1, \sigma_2)^\top : \Omega \to \mathbb{R}^2.$$

Let $p = 2$, $V = W_0^{1,2}(\Omega)$, $Q = L^2(\Omega)$, define $\varphi \in W^{1,2}(\Omega)$ as $\varphi = 0$, $g \in V^*$ as $g = 0$, and $f \in Q^*$ such that

$$\langle f, v \rangle_Q = \int_\Omega h(\boldsymbol{x}) v(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x}, \qquad \forall v \in V.$$

(a) Define, for this problem,

  i. the coefficient functions $a_\alpha(\boldsymbol{x}, \boldsymbol{\xi})$, $\boldsymbol{\xi} \in \mathbb{R}^\kappa$, for all multi-indices α, where $|\alpha| \leq 1$,
  ii. the divergence form of the Dirichlet problem,
  iii. the boundary value problem $(\mathcal{A}, V, Q)$,
  iv. the definition of the weak solution of the boundary value problem, and
  v. an operator $T : V \to V^*$ such that the set of solution of $Tu = 0$ is equivalent to the set of weak solutions to the boundary value problem.

(b) Derive conditions for μ, b, and h, such that Theorem 3.18 can be applied to show existence of a weak solution of the boundary value problem and any additional conditions necessary to ensure the weak solution is unique; i.e., state conditions such that

  i. T is monotone,
  ii. T is strictly monotone,
  iii. T is coercive, and
  iv. $a_\alpha \in \mathrm{CAR}^*(2)$, $|\alpha| \leq 1$

(c) In the case that $b(u) \equiv 0$ additionally state conditions on μ such that T is

  i. strongly monotone, and
  ii. Lipschitz continuous.