1. Prove only boundedness of Nemyckii operator for Theorem 3.11.

From assumption with $\xi = \delta_k u = (D^\beta u)_{|\beta|\le k}$

$$|f(x,\delta_k u)|^r \le \left| c\Big(\sum_{|\beta|<k-\frac{n}{p}} |D^\beta u|\Big)\right|^r \left| g(x) + \sum_{k-\frac{n}{p}\le|\beta|\le k} |D^\beta u|^{\frac{q(\beta)}{r}} \right|^r$$

$$\le 2^{r-1} \left| c\Big(\sum_{|\beta|<k-\frac{n}{p}} |D^\beta u|\Big)\right|^r \left(|g(x)|^r + \Big(\sum_{k-\frac{n}{p}\le|\beta|\le k} |D^\beta u|^{\frac{q(\beta)}{r}}\Big)^r \right)$$

by hint inequality. Apply same inequality recursively:

$$2^{r-1}\Big(\sum_{k-\frac{n}{p}\le|\beta|\le k} |D^\beta u|^{\frac{q(\beta)}{r}}\Big)^r \le (2^{r-1})^S \sum_{k-\frac{n}{p}\le|\beta|\le k} |D^\beta u|^{q(\beta)}$$

where $$S = \underbrace{\frac{(n+k)!}{n!\,k!}}_{\#\text{multiindices of size} \le k} - \underbrace{\frac{(n+k-\frac{n}{p}-1)!}{n!\,(k-\frac{n}{p}-1)!}}_{\#\text{multi-indices of size} < k-\frac{n}{p}}$$

$$\Rightarrow |f(x,\delta_k u(x))|^r \le C_1 \left| c\Big(\sum_{|\beta|<k-\frac{n}{p}} |D^\beta u|\Big)\right|^r \left(|g(x)|^r + \sum_{k-\frac{n}{p}\le|\beta|\le k} |D^\beta u|^{q(\beta)} \right)$$

By Theorem 3.8(iii) $|D^\beta u| \le C\|u\|_{k,p}$ for $|\beta|<k-\frac{n}{p}$;
then, $\exists$ constant C_2 (depending on $\|u\|_{k,p}$) such that

$$|f(x,\delta_k u)|^r \le C_1 C_2 \left(|g(x)|^r + \sum_{k-\frac{n}{p}\le|\beta|\le k} |D^\beta u|^{q(\beta)} \right)$$

Then, $\|\mathcal{N}(\delta_k u)\|_r = \int_\Omega |f(x,\delta_k u)|^r dx$

$$= C_1 C_2 \left(\int_\Omega |g(x)|^r + \sum_{k-\frac{n}{p}\le|\beta|\le k} \int_\Omega |D^\beta u|^{q(\beta)} \right)$$

$$= C_1 C_2 \left(\|g(x)\|_r^r + \sum_{k-\frac{n}{p}\le|\beta|\le k} \|D^\beta u\|_{q(\beta)}^{q(\beta)} \right)$$

As , by Theorem 3-8 (i) & (ii)

$$\|D^\beta u\|_{q(\beta)} \leq C\|u\|_{k,p}$$

$$\Rightarrow \|\mathcal{M}(\delta_k u)\|_r \leq C_1 C_2 \left(\|g(x)\|_r^r + C\|u\|_{k,p}^{q(k)}\right)$$

$$= M(\|u\|_{k,p})$$

where M is a function depending on $\|u\|_{k,p}$ $\square$

2. Prove Theorem 3.12

$$|\langle Au,v\rangle| \le \sum_{|\alpha|\le k} \int_\Omega |a_\alpha(x,\delta_k u)|\,|D^\alpha v|\,dx$$

$$\le \sum_{|\alpha|<k-\frac{n}{p}} \|a_\alpha(x,\delta_k u)\|_1 \|D^\alpha v\|_\infty$$

$$+ \sum_{k-\frac{n}{p}\le|\alpha|\le k} \|a_\alpha(x,\delta_k u)\|_{\frac{q(\alpha)}{q(\alpha)-1}} \|D^\alpha v\|_{q(\alpha)}$$

By Theorem 3.8 $\|D^\alpha v\|_{q(\alpha)} \le C\|u\|_{k,p}$ & $\|D^\alpha v\|_\infty \le C\|u\|_{k,p}$

$$\Rightarrow |\langle Au,v\rangle| \le C\Big(\sum_{|\alpha|<k-\frac{n}{p}} \|a_\alpha(x,\delta_k u)\|_1 + \sum_{k-\frac{n}{p}\le|\alpha|\le k} \|a_\alpha(x,\delta_k u)\|_{\frac{q(\alpha)}{q(\alpha)-1}}\Big)\|v\|_{k,p}$$

Note, by conditions we have that

$$|a_\alpha(x,\xi)| \le C\Big(\sum_{|\beta|<k-\frac{n}{p}} |\xi_\beta|\Big)\Big(g(x) + \sum_{k-\frac{n}{p}\le|\beta|\le k} |\xi_\beta|^{\frac{q(\beta)}{r}}\Big)$$

where $r = \begin{cases} \frac{q(\alpha)}{q(\alpha)-1} & |\alpha|\ge k-\frac{n}{p} \\ 1 & |\alpha|<k-\frac{n}{p} \end{cases}$

$\Rightarrow$ By Theorem 3.11 that $\|a_\alpha(x,\delta_k u(x))\|_r$ is bounded

$\Rightarrow |\langle Au,v\rangle| \le \sum_{|\alpha|\le k} M_\alpha(\|u\|)_{k,p}\|v\|_{k,p}$ where M_α function

Then, defining the norm for the dual space $(W^{k,p}(\Omega))^*$

$$\|Au\|_{k,p,*} = \sup_{v\in W^{k,p}} \frac{\langle Au,v\rangle}{\|v\|_{k,p}}$$

$$\le \sup_{v\in W^{k,p}} \sum_{|\alpha|\le k} M_\alpha(\|u\|_{k,p}) \Rightarrow A \text{ bounded.}$$

For continuity, as $u_n \to u$ we have that

$$\|Au - Au_n\|_{k,p,*} = \sup_{v \in W^{k,p}} \frac{\langle Au - Au_n, v \rangle}{\|v\|_{k,p}}$$

$$\leq \sum_{|\alpha| < k - \frac{n}{p}} \|a_\alpha(x, \delta_k u) - a_\alpha(x, \delta_k u_n)\|_1$$

$$+ \sum_{k - \frac{n}{p} \leq |\alpha| \leq k} \|a_\alpha(x, \delta_k u) - a_\alpha(x, \delta_k u_n)\|_{\frac{q(\alpha)}{q(\alpha)-1}}$$

By Theorem 3.11, with $r = \begin{cases} \frac{q(\alpha)}{q(\alpha)-1} & |\alpha| \geq k - \frac{n}{p} \\ 1 & |\alpha| < k - \frac{n}{p} \end{cases}$,

$\|a_\alpha(x, \delta_k u) - a_\alpha(x, \delta_k u_n)\| \to 0$ as Nemyckii operator for a_α is continuous

$\Rightarrow \|Au - Au_n\|_{k,p} \to 0$

$\Rightarrow Au \to Au_n$

$\Rightarrow A$ continuous.