1. Show that every monotone and potential operator is demicontinuous.

Let $A: X \to X^*$ be monotone; then, by Lemma 2.7(4) to show demicontinuous it is sufficient to show that

$$\langle f - Aw, u-w\rangle \geq 0 \ \forall w \in X \Rightarrow Au = f, \ f \in X^*$$

Let $w = u + t(v-u) \in X$, $\forall v \in X$; then,

$$-\langle f, t(v-u)\rangle + \langle Aw, t(v-u)\rangle \geq 0$$

$$\Rightarrow \qquad \langle f, t(v-u)\rangle \leq t\langle Aw, v-u\rangle$$
$$= t\langle Aw, v-w+t(v-u)\rangle$$
$$= t\langle Aw, (v+t(v-u))-w\rangle \quad (1)$$

If A is a potential operator; then, there exists a potential F with Gâteaux derivative $F' \equiv A$. By Theorem 2.23 A monotone $\Leftrightarrow F(y) \geq F(x) + \langle Ax, y-x\rangle \ \forall x, y \in X$

Therefore, from (1) (with $x = w$ and $y = v + t(v-u)$)

$$\langle f, t(v-u)\rangle \leq t\langle Aw, (v+t(v-u))-w\rangle$$
$$\leq t(F(v+t(v-u)) - F(w))$$

Divide by $t \neq 0$ and take limit as $t \to 0$

$$\langle f, v-u\rangle \leq \lim_{t\to 0} (F(v-t(v-u)) - F(u+t(v-u)))$$
$$= F(v) - F(u)$$

$$\Rightarrow F(u) - \langle f, u\rangle \leq F(v) - \langle f, v\rangle \qquad \forall v \in X$$

$$\Rightarrow F(u) - \langle f, u\rangle = \min_{v \in X} (F(v) - \langle f, v\rangle) \quad \text{as } u \in X$$

By Lemma 2.24 $\Rightarrow$ $Au = f$, $f \in X^*$ has a solution.

Hence, shown that $\langle f - Aw, u-w\rangle \geq 0 \ \forall w \in X \Rightarrow Au = f, f \in X^*$

$\Rightarrow A$ demicontinuous by Lemma 2.7(4).

2. Lemma 2.26 $A: X \to X^*$ potential monotone operator. Then, for $u \in X$ be solution of $Au = f$, $f \in X^*$, it is necessary and sufficient for

$$\int_0^1 \langle Atu, u\rangle dt - \langle f, u\rangle = \min_{v \in X}\left[\int_0^1 \langle Atv, v\rangle dt - \langle f, v\rangle\right]$$

Proof A monotone potential operator
$\Rightarrow$ A demicontinuous (Lemma 2.25)
$\Rightarrow$ A radially continuous (Lemma 2.7)

From Lemma 2.20

$$F(v) = F(0) + \int_0^1 \langle Atv, v\rangle dt \quad v \in X \quad (2)$$

By Lemma 2.24 it is necessary and sufficient for a solution to $Au = f$, $f \in X^*$ to exist for

$$F(u) - \langle f, u\rangle = \min_{v \in X}\left(F(v) - \langle f, v\rangle\right)$$

Substitute in (2):

$$F(0) + \int_0^1 \langle Atu, u\rangle dt - \langle f, u\rangle = \min_{v \in X}\left(F(0) + \int_0^1 \langle Atv, v\rangle dt - \langle f, v\rangle\right)$$

$$\Rightarrow \int_0^1 \langle Atu, u\rangle dt - \langle f, u\rangle = \min_{v \in X}\left(\int_0^1 \langle Atv, v\rangle dt - \langle f, v\rangle\right)$$

is necessary and sufficient for a solution to $Au = f$, $f \in X^*$ to exist. $\square$

3. Prove that for $A: X \to X^*$ with inverse $A^{-1}: X^* \to X$ that

A monotone potential operator $\Leftrightarrow$ A^{-1} monotone potential operator

It is sufficient to show that

A monotone potential operator $\Rightarrow$ A^{-1} monotone potential operator

due to symmetry of statement.

- If A is monotone then

$$\langle Au - Av, u - v \rangle \geq 0 \quad \forall u, v \in X$$

Let $u = A^{-1}u^*$ & $v = A^{-1}v^*$, $\forall u^*, v^* \in X^*$

$$\Rightarrow \langle u^* - v^*, A^{-1}u^* - A^{-1}v^* \rangle \geq 0 \quad \forall u^*, v^* \in X^*;$$

i.e; A^{-1} is monotone

- The potential F of A is a finite convex functional which is continuously Gâteaux differentiable and, thus, weakly lower semicontinuous

$$\Rightarrow \operatorname{grad} F^* = A^{-1} \quad \text{(Theorem 2.31)}$$

$\Rightarrow A^{-1}$ is a potential operator.

4. Theorem 2.33

Let $A: X \to X^*$ be a strictly monotone, coercive, potential operator. Then, there exists inverse $A^{-1}: X^* \to X$ which is a strictly monotone potential operator.

The functional

$$F(x) = \int_0^1 \langle Atx, x \rangle dt, \quad x \in X$$

is the potential of A and for any $x \in X$ and $x^* \in X^*$

$$F^*(x^*) = F^*(0) + \int_0^1 \langle x^*, A^{-1}tx^* \rangle dt$$

$$F^*(0) = -F(A^{-1}0)$$

$$F(x) + F^*(x^*) - \langle x^*, x \rangle \geq 0$$

$$F(x) + F^*(Ax) - \langle Ax, x \rangle = 0$$

where F^* is the potential of A^{-1}.

Proof

- From Corollary 2.32

  A strictly monotone potential operator $\Rightarrow A^{-1}$ monotone potential operator

  Define $y^* \neq x^* \in X^*$, $x = A^{-1}x^*$, $y = A^{-1}y^*$

  $$\langle x^* - y^*, A^{-1}x^* - A^{-1}y^* \rangle = \langle Ax - Ay, x - y \rangle > 0$$

  $\Rightarrow A^{-1}$ strictly monotone

- Consider potential F for A and a sequence $\{x_n\}$ which weakly converges to x_0. Define $f = Ax_0$, then by Lemma 2.26 the existence of solution x_0 to Ax_0 equivalent to

$$F(x_0) - \langle Ax_0, x_0 \rangle = \min_{v \in X} (F(v) - \langle Ax_0, v \rangle)$$

$$\leq \liminf_{n \to \infty} (F(x_n) - \langle Ax_0, x_n \rangle)$$

$$\Rightarrow F(x_0) - \langle Ax_0, x_0 \rangle \le \liminf_{n\to\infty} F(x_n) - \langle Ax_0, x_0 \rangle$$

$$\Rightarrow \quad F(x_0) \le \liminf_{n\to\infty} F(x_n)$$

$\Rightarrow$ F weakly lower semicontinuous

- From Corollary 2.28 $F(x) = \int_0^1 \langle Atx, x \rangle dt$ potential of A
- From Theorem 2.31 F^* potential of A^{-1} and
  $F(x) + F^*(Ax) - \langle Ax, x \rangle = 0 \Rightarrow F^*(0) = -F(A^{-1}0)$
- By Lemma 2.29 $F(x) + F^*(x^*) \ge \langle x^*, x \rangle$
- By Lemma 2.20 $F(x) = F(0) + \int_0^1 \langle Atx, x \rangle dt$

5. <u>Corollary 2.34</u> $A: X \to X^*$ strictly monotone, coercive, potential operator with potential F. For any $f \in X^*$ $\exists!$ solution $u \in X$ of $Au = f$ which minimises the potential of the problem $G = F - f$ and

$$G(u) = F(u) - \langle f, u \rangle = \min_{v \in X} \left[\int_0^1 \langle Atv, v \rangle dt - \langle f, v \rangle \right]$$

$$= -\int_0^1 \langle f, A^{-1} tf \rangle dt + \int_0^1 \langle AtA^{-1}0, A^{-1}0 \rangle dt$$

As A strictly monotone coercive potential operator

$\Rightarrow Au = f$ has unique solution (Th 2.27)

Consider

$$
\begin{aligned}
G(u) &\equiv F(u) - \langle f, u \rangle \\
&= \int_0^1 \langle Atu, u \rangle dt - \langle f, u \rangle && \text{(Th 2.33)} \\
&= \min_{v \in X} \left[\int_0^1 \langle Atv, v \rangle dt - \langle f, v \rangle \right] && \text{(Th 2.26)} \\
&= \min_{v \in X} \left(F(v) - \langle f, v \rangle \right) && \text{(Th 2.33)}
\end{aligned}
$$

which proves u minimises G

$$
\begin{aligned}
G(u) &\equiv F(u) - \langle f, u \rangle = F(u) - \langle Au, u \rangle \\
&= -F^*(Au) = -F^*(f) && \text{(Th 2.33)} \\
&= -F^*(0) - \int_0^1 \langle f, A^{-1} tf \rangle dt && \text{(Th 2.33)} \\
&= F(A^{-1}0) - \int_0^1 \langle f, A^{-1} tf \rangle dt && \text{(Th 2.33)} \\
&= \int_0^1 \langle AtA^{-1}0, A^{-1}0 \rangle dt - \int_0^1 \langle f, A^{-1} tf \rangle dt && \text{(Th 2.33)}
\end{aligned}
$$

$\square$