Nonlinear Differential Equations

Practical Exercises 5

Due: 27th March 2024

1. Show that every monotone and potential operator is demicontinuous (Lemma 2.25).

2. Prove the following.

   Lemma 2.26. *Let $A : X \to X^*$ be a monotone potential operator. Then, for $u \in X$ to be a solution of $Au = f$, $f \in X^*$, it is necessary and sufficient for*

   $$\int_0^1 \langle Atu, u\rangle \,\mathrm{d}t - \langle f, u\rangle = \min_{v\in X}\left[\int_0^1 \langle Atv, v\rangle \,\mathrm{d}t - \langle f, v\rangle\right].$$

3. Prove that for an operator $A : X \to X^*$ with inverse $A^{-1} : X^* \to X$ that

   $$A \text{ monotone potential operator} \iff A^{-1} \text{ monotone potential operator.}$$

   (Corollary 2.32).

4. Prove the following.

   Theorem 2.33. *Let $A : X \to X^*$ be a strictly monotone, coercive, potential operator. Then, there exists an inverse operator A^{-1}, which is a strictly monotone potential operator. The functional F,*

   $$F(x) = \int_0^1 \langle Atx, x\rangle \,\mathrm{d}t, \qquad x \in X,$$

   is the potential of A and for any $x \in X$ and $x^* \in X$

   $$\begin{aligned} F^*(x^*) &= F^*(0) + \int_0^1 \langle x^*, A^{-1}tx^*\rangle \,\mathrm{d}t,\\ F^*(0) &= -F(A^{-1}0),\\ F(x) + F^*(x^*) - \langle x^*, x\rangle &\geq 0,\\ F(x) + F^*(Ax) - \langle Ax, x\rangle &= 0, \end{aligned}$$

   where F^* is the potential of A^{-1}.

5. Prove the following.

   Corollary 2.34. *Let $A : X \to X^*$ be a strictly monotone, coercive, potential operator with potential F. For any $f \in X^*$ there exists a unique solution $u \in X$ of $Au = f$ which minimises the potential of the problem $G = F - f$ and*

   $$\begin{aligned} G(u) &\equiv F(u) - \langle f, u\rangle\\ &= \min_{v\in X}\left(\int_0^1 \langle Atv, v\rangle \,\mathrm{d}t - \langle f, v\rangle\right)\\ &= -\int_0^1 \langle f, A^{-1}tf\rangle \,\mathrm{d}t + \int_0^1 \langle AtA^{-1}0, A^{-1}0\rangle \,\mathrm{d}t. \end{aligned}$$