1. $A: \mathbb{R}^m \to \mathbb{R}^m$, $m > 0$, be SPD.

a) Show that $T: \mathbb{R}^m \to \mathbb{R}^m$, $v \mapsto Av$ is strongly continuous and Lipschitz continuous.

A symmetric $\Rightarrow$ $A = Q \Lambda Q^T$, where Q orthogonal and Λ diagonal with $\Lambda_{ii} = \lambda_i$, where λ_i is ith eigenvalue of A.

A is SPD:

For eigenvalue λ_i with eigenvector x_i:

$$0 < x_i^T (A x_i) = x_i^T (\lambda_i x_i) = \lambda_i x_i^T x_i = \lambda_i \|x_i\|$$

As $\|x_i\| \geq 0$ $\Rightarrow$ $\lambda_i > 0$ $\forall \lambda_i \in \sigma(A)$

So all eigenvalues are strictly positive $\Rightarrow$ $0 < \min_{\lambda \in \sigma(A)} \lambda$

- Strongly monotone:

$$\begin{aligned}
(T(u) - T(v), u - v) &= (Au - Av)^T (u - v) \\
&= (u-v)^T A^T (u-v) \\
&= (u-v)^T Q^T \Lambda Q (u-v) \\
&= \sum_{i=1}^m ((u-v)^T Q^T)_i \lambda_i (Q(u-v))_i \\
&\geq \min_{\lambda \in \sigma(A)} \lambda \sum_{i=1}^m ((u-v)^T Q^T)_i (Q(u-v))_i \\
&= \underbrace{\min_{\lambda \in \sigma(A)}}_{=:M} (u-v)^T \underbrace{Q^T Q}_{= I} (u-v) = M \|u-v\|^2
\end{aligned}$$

- Lipschitz continuous:

$$\begin{aligned}
\|T(u) - T(v)\|^2 = (Au - Av)^T (Au - Av) &= (u-v)^T Q^T \Lambda Q (Au - Av) \\
&\leq \underbrace{\max_{\lambda \in \sigma(A)}}_{L} \lambda (u-v)^T Q^T Q (Au - Av) \\
&= L \langle T(u) - T(v), u - v \rangle \\
&\leq L \|u - v\| \|Au - Av\|
\end{aligned}$$

Divide by $\|T(u) - T(v)\|$ to complete proof.

b) $b \in \mathbb{R}^m$. Show $\exists \delta \in \mathbb{R}, \delta > 0$ such that

$$x_{n+1} = x_n - \delta(Ax_n - b), \quad n \geq 0$$

converges to $A^{-1}b$ for any starting vector $x_0 \in \mathbb{R}^m$.

Define $\tilde{T}(x) = x - \delta(T(x) - b)$, $\delta > 0$; then, as T strongly monotone and Lipschitz continuous

$$\|\tilde{T}(x) - \tilde{T}(y)\|^2 = \|x-y\|^2 - 2\delta(T(x) - T(y), x-y) + \delta^2 \|T(x) - T(y)\|^2$$

$$\leq \|x-y\|^2 - 2\delta M \|x-y\|^2 + L^2\delta^2 \|x-y\|^2$$

$$= (1 - 2\delta M + L^2\delta^2)\|x-y\|^2$$

Select δ such that $2M > L^2\delta$; then, $\tilde{T}$ is strongly contractive; therefore, by Banach F.P. $\exists! \bar{x}$ such that

$$\bar{x} = \tilde{T}(\bar{x}) \Rightarrow A\bar{x} = b \Rightarrow \bar{x} = A^{-1}b$$

and the iteration

$$x_{n+1} = \tilde{T}(x_n) = x_n - \delta(Ax_n - b)$$

converges to $\bar{x} = A^{-1}b$.

2. Consider $X=\ell^2$ and $A: X \to X$ as $Ax=y$,
$x=(\xi_1, \dots \xi_k, \dots)$, $y=((\xi_1)^1, \dots, (\xi_k)^k, \dots)$
Show A is continuous and not bounded.

Select $x^{(n)}=(\xi_i^{(n)}) \in X$ such that $x^{(n)} \to x$, $x=(\xi_i)$
The sequence is bounded; i.e, $\exists C>0$ such that
$\|x^{(n)}\| \leq C$ and $\|x\| \leq C$. From convergence and definition of norm in X it follows that $\exists N_0, n_0$ such that for all $i \geq N_0$ and $n \geq N_0$

$$|\xi_i^{(n)}| < \frac{1}{2} \quad \text{and} \quad |\xi_i| < \frac{1}{2}.$$

Then, for $n \geq n_0$

$$\|Ax^{(n)} - Ax\|^2 = \sum_{i=0}^{N_0} \left((\xi_i^{(n)})^i - \xi_i^i\right)^2 + \sum_{i=N_0+1}^{\infty} \left((\xi_i^{(n)})^i - \xi_i^i\right)^2$$
$$\leq C_1 \sum_{i=0}^{N_0} (\xi_i^{(n)} - \xi_i)^2 + C_2 \sum_{i=N_0+1}^{\infty} (\xi_i^{(n)} - \xi_i)^2$$
$$\leq C \|x^{(n)} - x\|^2 \to 0$$

where
$$C_1 = \max_{i=1,\dots,N_0} (iC^{i-1})^2, \qquad C_2 = \max_{i=N_0+1,\dots,\infty} \left(i\left(\tfrac{1}{2}\right)^{i-1}\right)^2$$

and $C=\max(C_1, C_2)$ which shows continuity.

Now define $x^{(n)}=(\xi_i^{(n)})$, $\xi_n^{(n)}=2$ & $\xi_i^{(n)}=0$ for $i \neq n$
Then, $\|x^{(n)}\|=2$, $\|Ax^{(n)}\|=2^n$
Clearly, there exists no function $M_1: \mathbb{R}_+ \to \mathbb{R}_+$ such that
$$\|Au\| \leq M_1(\|u\|)$$
as for such a function to exist
$$\|Ax^{(n)}\| = 2^n \leq M(\|x^{(n)}\|) = M(2) \quad \forall n \in \mathbb{N}$$
which is not possible; therefore, A is not bounded.

3. a) Show A uniformly monotone $\Rightarrow$ A strictly monotone

Assume $u \neq v$, then $\|u-v\| \neq 0$ and therefore $a(\|u-v\|) > 0$

$$\Rightarrow \langle Au - Av, u-v \rangle \geq a(\|u-v\|)\|u-v\| > 0$$

b) A strictly monotone $\Rightarrow$ A monotone

$$\left.\begin{array}{l} \text{If } u \neq v \text{ then } \langle Au-Av, u-v\rangle > 0 \\ \text{If } u = v \text{ then } \langle Au-Av, u-v\rangle = 0 \end{array}\right\} \Rightarrow \text{monotone}$$

c) A α-monotone $\Rightarrow$ A monotone

$$\langle Au-Av, u-v\rangle \geq (\alpha(\|u\|)-\alpha(\|v\|))(\|u\|-\|v\|)$$

- If $\|u\|>\|v\|$ then $\alpha(\|u\|) > \alpha(\|v\|)$ as α strictly inc.

$$\Rightarrow \|u\|-\|v\|>0 \text{ and } \alpha(\|u\|)-\alpha(\|v\|) > 0$$

$$\Rightarrow (\alpha(\|u\|)-\alpha(\|v\|))(\|u\|-\|v\|) > 0$$

- If $\|u\|<\|v\|$ then $\alpha(\|u\|) < \alpha(\|v\|)$

$$\Rightarrow \|u\|-\|v\|<0 \text{ and } \alpha(\|u\|)-\alpha(\|v\|) < 0$$

$$\Rightarrow (\alpha(\|u\|)-\alpha(\|v\|))(\|u\|-\|v\|) > 0$$

- If $\|u\|=\|v\|$ then $\|u\|-\|v\|=0$

$$\Rightarrow (\alpha(\|u\|)-\alpha(\|v\|))(\|u\|-\|v\|) = 0$$

d) A strongly monotone & B strongly monotone $\Rightarrow$ $A+B$ strongly monotone

A strongly monotone: $\exists M_A: \langle Au-Av, u-v\rangle \geq M_A\|u-v\|$

B strongly monotone: $\exists M_B: \langle Bu-Bv, u-v\rangle \geq M_B\|u-v\|$

$$\langle (A+B)u-(A+B)v, u-v\rangle = \langle Au-Av, u-v\rangle + \langle Bu-Bv, u-v\rangle$$

$$\geq \underbrace{(M_A+M_B)}_{M}\|u-v\|^2$$

e) A strongly monotone & B monotone $\Rightarrow$ $A+B$ strongly monotone

$$\langle (A+B)u-(A+B)v, u-v\rangle = \langle Au-Av, u-v\rangle + \underbrace{\langle Bu-Bv, u-v\rangle}_{\geq 0}$$

$$\geq \langle Au-Av, u-v\rangle$$

$$\geq M_A \|u-v\|^2$$