Nonlinear Differential Equations

Practical Exercises 3

Due: 13th March 2024

1. Let $\boldsymbol{A} : \mathbb{R}^m \to \mathbb{R}^m$, $m > 0$, be a symmetric positive definite matrix.
   (a) Show that the operator
   $$\begin{aligned} T : \mathbb{R}^m &\to \mathbb{R}^m, \\ v &\mapsto \boldsymbol{A}v, \end{aligned}$$
   is strongly monotone and Lipschitz continuous.
   Hint. Consider the eigendecomposition of $\boldsymbol{A}$.
   (b) Given $\boldsymbol{b} \in \mathbb{R}^m$, show that there exists a positive constant $\delta \in \mathbb{R}$ such that the iteration
   $$\boldsymbol{x}_{n+1} = \boldsymbol{x}_n - \delta(\boldsymbol{A}\boldsymbol{x}_n - \boldsymbol{b}), \qquad n \geq 0,$$
   converges to $\boldsymbol{A}^{-1}\boldsymbol{b}$ for any starting vector $\boldsymbol{x}_0 \in \mathbb{R}^m$.

2. Continuous linear operators are *always* bounded; whereas, continuous *nonlinear* operators may not be bounded.

   For example, consider $X := \ell^2$ and define the operator $A : X \to X$ as
   $$Ax = y, \qquad x = \{\xi_1, \dots, \xi_k, \dots\}, y = \{(\xi_1)^1, \dots, (\xi_k)^k, \dots\}.$$
   Show that A is continuous but not bounded.

   Hint. For continuity construct a (bounded) convergent sequence. You can also use the trivial statements
   $$(a^i - b^i)^2 \leq (a-b)^2 i r^{i-1}, \qquad \text{for } a \geq 0, b \geq 0, i \in \mathbb{N}, r = \max(a, b)$$
   and
   $$\lim_{i\to\infty} \left(i \left(\frac{1}{2}\right)^{i-1} \right) \to 0$$
   without proof.

3. Show that for the operators $A, B : X \to X^*$, on a Banach space X, that
   (a) A uniformly monotone $\implies$ A strictly monotone
   (b) A strictly monotone $\implies$ A monotone
   (c) A α-monotone $\implies$ A monotone
   (d) A strongly monotone and B strongly monotone $\implies$ $A + B$ strongly monotone
   (e) A strongly monotone and B monotone $\implies$ $A + B$ strongly monotone