Exercise 1 - Solutions

1. By means of suitable counterexamples show that all assumptions of the contraction mapping theorem are necessary.

Note closed subset of complete metric space is also complete. Show counterexamples to M being complete & $T: M \to M$ strongly contractive:

- Strongly contractive:

Consider $f(x) = \sqrt{1+x^2}$

$$f'(x) = \frac{x}{\sqrt{1+x^2}} < 1 \quad \text{for } \forall x \in [0,\infty)$$

$$\lim_{x \to \infty} f'(x) = 1$$

So $|f(x) - f(y)| = f'(\xi)|x-y| < |x-y| \quad x \neq y$

by mean value theorem

or $|f(x) - f(y)| \leq k|x-y|$ with $k=1$

$\Rightarrow$ <u>not</u> strongly contractive

Clearly $f(x)$ has no fixed point as

$$f(x) = \sqrt{1+x^2} > x \quad \forall x \in \mathbb{R}$$

- Not complete: $X = \mathbb{R} \setminus \left\{\frac{b}{1-a}\right\}$, $a,b \in \mathbb{R}$, $0<a<1$

and $d(x,y) = |x-y|$

$$h: X \to X$$
$$x \mapsto ax+b$$

If F.P. $\exists x \in X$ s.t. $h(x) = ax+b = x$

$\Rightarrow ax+b = x \Rightarrow b = x - ax$

$\Rightarrow x = \frac{b}{1-a} \notin X \Rightarrow h(x)$ has no F.P.

2. $a>0$ consider

$$u(x) = 1 + \frac{1}{\pi}\int_{-a}^{a} \frac{u(y)}{1+(x-y)^2}\,dy \qquad -a \le x \le a$$

Prove, this has unique continuous solution for any $a \in (0,\infty)$.

Consider $T(u) = 1 + \frac{1}{\pi}\int_{-a}^{a} \frac{u(y)}{1+(x-y)^2}\,dy$

Looking for FP of T: $u(x) = T(u(x))$, $x \in [-a,a]$

Define $f(u(y),x,y) = \frac{1}{\pi}\frac{u(y)}{1+(x-y)^2}$

$$|f(u(y),x,y) - f(\tilde{u}(y),x,y)| = \frac{1}{\pi}\left|\frac{u(y)-\tilde{u}(y)}{1+(x-y)^2}\right|$$

$$\le \frac{1}{\pi|1+(x-y)^2|}|u(y)-\tilde{u}(y)|$$

$$\le \frac{1}{\pi}|u(y)-\tilde{u}(y)|$$

- Assume f continuous on $Q = \{|y|\le a, |x|\le a, |u-1|\le b\}$

Define $K = \sup_{(u,x,y)\in Q} |f(u(y),x,y)| < \infty$ $\qquad b \ge 2aK$

- Define $Z = \{v:[-a,a] \to \mathbb{R} \text{ continuous}\}$

& introduce norms $\|v\|_0 = \max_{x\in[-a,a]} |v(x)|$

$$\|v\|_1 = \max_{x\in[-a,a]} e^{-|x|}|v(x)|$$

which are clearly equivalent. Hence,

$(Z, \|\cdot\|_0)$ & $(Z, \|\cdot\|_1)$ are both complete

- Define $M := \{v \in Z, \|v-1\|_0 \le b\}$ & consider

$T: (M, \|\cdot\|_1) \to (Z, \|\cdot\|_1)$

Consider assumptions of Banach F.P.

i) M closed follows as before

ii) $T: M \Rightarrow M$

Let $u \in M$

$$\|T(u(x)) - 1\|_0 = \max_{x\in[-a,a]} \left| \frac{1}{\pi} \int_{-a}^{a} \frac{u(y)}{1+(x-y)^2} dy \right|$$

$$\leq \max_{x\in[-a,a]} \int_{-a}^{a} \left| \frac{1}{\pi} \frac{u(y)}{1+(x-y)^2} \right| dy$$

$$= \max_{x\in[-a,a]} \int_{-a}^{a} |f(u,x,y)| dy$$

$$\leq 2aK \leq b$$

iii) $$\|T(u) - T(\tilde{u})\|_1 = \left\| \int_{-a}^{a} f(u(y),x,y) - f(\tilde{u}(y),x,y) dy \right\|_1$$

$$\leq \max_{x\in[-a,a]} e^{-|x|} \int_{-a}^{a} \frac{1}{\pi} |u(y) - \tilde{u}(y)| dy$$

$$= \max_{x\in[-a,a]} e^{-|x|} \frac{1}{\pi} \int_{-a}^{a} |u(y) - \tilde{u}(y)| e^{-|y|} e^{|y|} dy$$

$$\leq \max_{x\in[-a,a]} e^{-|x|} \frac{1}{\pi} \|u - \tilde{u}\|_1 \int_{-a}^{a} e^{|y|} dy$$

$$= \max_{x\in[-a,a]} 2e^{-|x|} \frac{1}{\pi} \|u - \tilde{u}\|_1 (e^{a} - 1)$$

$$= \frac{2}{\pi} (1 - e^{-a}) \|u - \tilde{u}\|_1$$

$$= k \|u - \tilde{u}\|_1 \text{ where } k = 1 - e^{-a} < 1$$

Kakutani's Counter-example

- Brouwer's F.P. does not hold in infinite dimensions

Let X be real (infinite dimensional) Hilbert space, and $\{\phi_i\}_{i=1}^{\infty} \subset X$ be an orthonormal system; i.e.,

$$(\phi_i, \phi_j)_X = \delta_{ij} \quad i,j \geq 1$$

Define the (separable) subspace

$$\tilde{X} := \left\{ x \in X : \exists \{\alpha_i\}_{i=1}^{\infty} \subset \mathbb{R} \text{ with } \|x - \sum_{i=1}^{\infty} \alpha_i \phi_i\|_X < \varepsilon,\ \varepsilon > 0 \right\}$$

$\Rightarrow x \in \tilde{X}$ has unique representation of the form

$$x = \sum_{i=1}^{\infty} \alpha_i \phi_i$$

Define the continuous mapping

$$F : X \to \tilde{X}, \quad F(x) := \sum_{i=1}^{\infty} \alpha_i \phi_{i+1}$$

If $\quad \underbrace{\|x\|_X}_{= \left(\sum_{i=1}^{\infty} \alpha_i^2\right)^{1/2}} = r \quad \Rightarrow \quad \underbrace{\|F(x)\|_X}_{= \left(\sum_{i=1}^{\infty} \alpha_i^2\right)^{1/2}} = r$

Furthermore, define $f : \tilde{X} \to \tilde{X}$

$$x \mapsto \tfrac{1}{2}(1 - \|x\|_X)\phi_i + F(x)$$

Define $B := \{x \in \tilde{X} : \|x\|_X \leq 1\}$. Then, for $x \in B$

$$\|f(x)\|_X \leq \tfrac{1}{2}(1 - \|x\|_X)\underbrace{\|\phi_1\|_X}_{=1} + \underbrace{\|F(x)\|_X}_{=\|x\|_X}$$

$$= \tfrac{1}{2}(1 - \underbrace{\|x\|_X}_{\leq 1})$$

$$\leq 1$$

$\Rightarrow f : B \to B$ continuously defined

Suppose f has a fixed point $\xi \in B$. Then,

$$f(\xi) = \xi \implies \xi - F(\xi) = f(\xi) - F(\xi) = \frac{1}{2}(1 - \|\xi\|_X)\phi_1$$

(i) $\xi = 0 \implies 0 - \underbrace{F(0)}_{=0} = \frac{1}{2}\phi_i \to$ Contradiction

(ii) $\|\xi\|_X = 1$. With $\xi = \sum_{i=1}^{\infty} \alpha_i \phi_i$ we conclude that

$$1 = \|\xi\|_X = \sum_{i=1}^{\infty} \alpha_i^2 = 1$$

However, $\xi - F(\xi) = 0$:

$$\sum_{i=1}^{\infty} \alpha_i \phi_i - \sum_{i=1}^{\infty} \alpha_i \phi_{i+1} = 0$$

$$\implies \alpha_1 \phi_1 + \sum_{i=2}^{\infty} (\alpha_i - \alpha_{i-1}) \phi_i = 0$$

$$\implies \alpha_1 = 0 \ \& \ \alpha_i - \alpha_{i-1} = 0 \ \forall i \geq 2$$

$$\implies \alpha_i = 0 \ \forall i \geq 1 \to \text{Contradiction}$$

(iii) $0 < \|\xi\|_X < 1$. As $\xi - F(\xi) = \frac{1}{2}(1 - \|\xi\|_X)\phi_1$

$$\sum_{i=1}^{\infty} \alpha_i \phi_i - \sum_{i=1}^{\infty} \alpha_i \phi_{i+1} = \underbrace{\frac{1}{2}(1 - \|\xi\|_X)}_{=: C \text{ where } 0 < C < 1} \phi_1$$

$$\implies \alpha_1 \phi_1 + \sum_{i=1}^{2} (\alpha_i - \alpha_{i-1}) \phi_i = C\phi_1$$

$$\implies \alpha_1 = C \ \& \ a_i - \alpha_{i-1} = 0 \ \forall i \geq 2$$

$$\implies \alpha_i = C > 0 \ \forall i \geq 1$$

Then, $\|\xi\|_X = \sum_{i=1}^{\infty} \alpha_i^2 = \sum_{i=1}^{\infty} \underbrace{C}_{>0} = \infty > 1 \to$ Contradiction