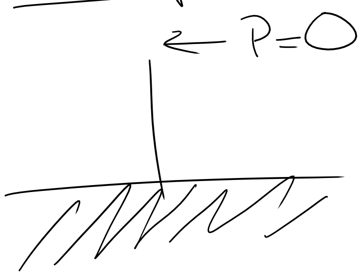
Motivation

Example - Bending rod with perpendicular load (Böhmer



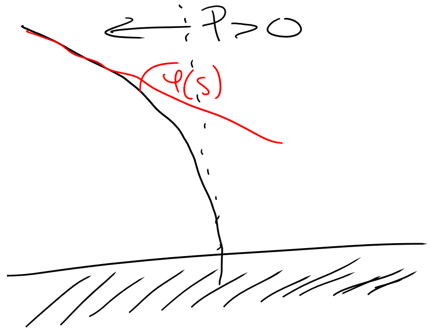
- Vertical rod clamped at lower end & free at top
- Load P applied to free end, perpendicular to rod's initial state.

- Displacement for small P is proportional to P; i.e., a linear relationship

$$x = cP \text{, for appropriate constant } c,$$

holds.

- Simple experiment shows rod will deform nonlinearly, or even break, when P passes a critical value.
- Need to build nonlinear model to incorporate these effects.



- We can obtain local strain energy, U, of the rod at a point with arc length s analogously to the kinetic energy of a moving body

$$U = \frac{\alpha}{2}\left(\frac{d\varphi}{ds}(s)\right)^2$$

where $\varphi(s)$ is the angle between the vertical & the tangent of the rod at s, and α is the bending stiffness of the material.

- Total energy of deformed rod is

$$U_B = \frac{\alpha}{2}\int_0^L \left(\frac{d\varphi}{ds}\right)^2 ds \, , \quad \varphi \in C^1[0,L]$$

for total rod length L.

- Potential energy, due to moving top of rod, is given by $-Px(L)$, where $x(L)$ is displacement at L.

Single trigonometry gives

$$\frac{dx}{ds} = \sin(\varphi(s))$$

so by fundamental theorem of calculus

$$x(L) = \int_0^L \sin \varphi(s)\, ds$$

- Total (potential) energy is sum of these terms

$$V(\varphi) := \frac{\alpha}{2}\int_0^L \left(\frac{d\varphi}{ds}\right)^2 ds - P\int_0^L \sin \varphi\, ds, \quad \varphi \in C^1[0,L]$$

- One of the fundamental principals of mechanics states an equilibrium of a system is characterised by a minimum of its potential; i.e, for every small ψ

$$V(\varphi+\psi) \geq V(\varphi)$$

- We claim this minimising function φ is characterised by a nonlinear boundary value problem

$$G(\varphi,\lambda) = \frac{d^2\varphi}{ds^2} + \lambda \cos\varphi = 0, \quad \lambda := \frac{P}{\alpha}$$

with boundary conditions

$$\varphi(0) = \frac{d\varphi}{ds}(L) = 0$$

(see Bühner for proof that solution minimises V).

- We can show that for small φ the linear & nonlinear models are related. For small φ we note that $\sin\varphi \approx \varphi$, $\cos\varphi \approx 1$

$$\Rightarrow \frac{d^2\varphi}{ds^2} + \lambda\cos\varphi \approx \frac{d^2\varphi}{ds^2} + \lambda = 0$$

$$\Rightarrow \varphi \approx -\frac{\lambda s^2}{2} + \mu s + \nu$$

From BC $v=0$ and $\mu = L\lambda = \frac{LP}{\alpha}$

Also, $0 \le \varphi(s) = \lambda\left(-\frac{s^2}{2} + Ls\right) \le \varphi(L) = \frac{PL^2}{2\alpha} \ll 1$

and $x(L) = \int_0^L \sin\varphi \, ds \approx \int_0^L \varphi \, ds = \lambda\left[-\frac{s^3}{6} + L\frac{s^2}{2}\right]_0^L$

$= P\frac{L^3}{3\alpha} = cP$

FP Examples

a) Algebraic systems: $F : \mathbb{R}^n \to \mathbb{R}^n$ (possibly nonlinear).
Then, find $\bar{x} \in \mathbb{R}^n$ s.t. $F(\bar{x}) = 0$. Possible formulations

(i) Linear relaxation

$$x = \underbrace{x - \omega F(x)}_{T(x)}, \quad \omega \in \mathbb{R}$$

(ii) Newton's method

$$x = x - \underbrace{DF}_{\text{Jacobian of } F}(x)^{-1} F(x) \quad \exists DF(x)^{-1}$$

b) Initial value problems: Find $y : [t_0, T] \to \mathbb{R}^n$ continuously differentiable such that

$$y'(t) = f(y(t), t) \quad t \in [t_0, T]$$
$$y(t_0) = y_0 \in \mathbb{R}^n$$

Possible fixed point iteration

$$y(t) = \underbrace{y_0 + \int_{t_0}^t f(y(\tau), \tau) \, d\tau}_{T(y)}$$

c) (Semilinear) PDE: Consider finding

$$u : \mathbb{R}^n \supset \Omega \to \mathbb{R}$$
$$(x_1, \dots, x_n) \mapsto u(x_1, \dots, x_n)$$

such that $\Delta u = f(u)$ in Ω, $\quad \Delta u = \sum_{i=1}^n \frac{\partial^2 u}{\partial x_i}$

↳ possibly nonlinear

with BCs $u|_{\partial\Omega}=0$

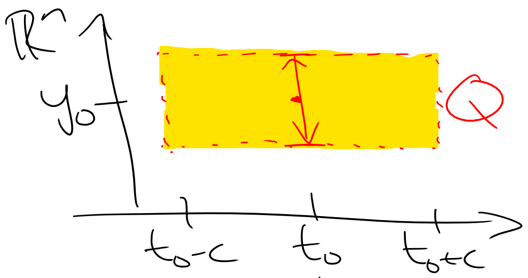
Possible fixed point iteration $u=\Delta^{-1}(f(u))$
where for given ϕ we define $\Delta^{-1}\phi$ by solution of linear problem
$$\begin{aligned}\Delta v &= \phi \text{ in } \Omega \\ v&=0 \text{ on } \partial\Omega\end{aligned} \iff v=\Delta^{-1}\phi$$

IVP Existence Solution

Consider $y'(t)=f(y(t),t) \quad t\in(t_0-c,t_0+c),\ c>0$
with initial condition $y(t_0)=y_0\in\mathbb{R}^n$
$$y:(t_0-c,t_0+c)\longrightarrow\mathbb{R}^n$$
$$f:\mathbb{R}^n\times(t_0-c,t_0+c)\longrightarrow\mathbb{R}^n$$
Consider FP by means of integration
$$y(t)=y_0+\int_{t_0}^{t} f(y(s),s)\,ds$$

Assumptions:

a) f is continuous on $Q=\{|t-t_0|\le a, |y-y_0|\le b\}$, where a,b are positive constants



b) f is Lipschitz continuous in y:
$$|f(y,s)-f(\tilde{y},s)|\le L|y-\tilde{y}| \quad \forall (y,s),(\tilde{y},s)\in Q$$

c) Define $K:=\sup_{(y,t)\in Q}|f(y,t)|<\infty$

Show:

a) There exists a unique solution y, which is continuous & exists in $t_0 - c \le t \le t_0 + c$ where $c = \min(a, \frac{b}{K})$

b) The sequence

$$y_{n+1}(t) = y_0 + \int_{t_0}^{t} f(y_n(s), s)\,ds, \quad n \geq 0$$

converges uniformly on $[t_0 - c, t_0 + c]$ to y.

Setup:

- Define space $Z := \{v : [t_0 - c, t_0 + c] \to \mathbb{R}^n \text{ continuous}\}$ and introduce the norms

$$\|v\|_0 = \max_{t \in [t_0 - c, t_0 + c]} |v(t)|$$

$$\|v\|_1 = \max_{t \in [t_0 - c, t_0 + c]} e^{-L|t - t_0|} |v(t)|$$

Z with norm $\|v\|_0$ is complete.

Moreover, $e^{-cL}\|v\|_0 \le \|v\|_1 \le \|v\|_0 \quad \forall v \in Z$

$\Rightarrow$ Z also complete w.r.t $\|\cdot\|_1$

- Define $M = \{v \in Z, \|v - y_0\|_0 \le b\}$ and consider the mapping $T : (M, \|\cdot\|_1) \to (Z, \|\cdot\|_1)$

$$y \mapsto T(y)(t) = y_0 + \int_{t_0}^{t} f(y(s), s)\,ds$$

Check assumptions of Banach F.P.

i) M is closed:

$\{y_n\} \subset M$ sequence converges to $y_\infty \in Z$ w.r.t $\|\cdot\|_1$

$$\lim_{n\to\infty} \|y_n - y_\infty\|_1 = 0 \Rightarrow \lim_{n\to\infty} \|y_n - y_\infty\|_0 = 0$$

$$y_n \in M: \|y_n - y_0\| \le b \quad \forall n \in \mathbb{N}$$

$$\Rightarrow y_\infty = \lim_{n\to\infty} y_n \in M \Rightarrow \text{closed}$$

ii) $T: M \to M$: Let $y \in M$

$$\|T(y) - y_0\|_0 = \max_{t\in[t_0-c, t_0+c]} \left| \int_{t_0}^{t} f(y(s), s)\,ds \right|$$

$$\le \max_{t\in[t_0-c,t_0+c]} \int_{t_0}^{t} \underbrace{|f(y(s),s)|}_{\le K} ds$$

$$\le Kc \le b$$

$$\Rightarrow T(y) \in M$$

iii) T is strongly contractive: $y, \tilde{y} \in M$

$$\|T(y) - T(\tilde{y})\|_1 = \left\| \int_{t_0}^{t} f(y(s), s) - f(\tilde{y}(s), s)\,ds \right\|_1$$

$$= \max_{t\in[t_0-c,t_0+c]} e^{-L|t-t_0|} \int_{t_0}^{t} \underbrace{|f(y(s),s) - f(\tilde{y}(s),s)|}_{\le L|y(s) - y(\tilde{s})|} ds$$

$$\le \max_{t\in[t_0-c,t_0+c]} e^{-L|t-t_0|} L \int_{t_0}^{t} \underbrace{|y(s) - \tilde{y}(s)| e^{-L|s-t_0|}}_{\le \|y-\tilde{y}\|_1} e^{L|s-t_0|} ds$$

$$\le \max_{t\in[t_0-c,t_0+c]} L\|y - \tilde{y}\|_1 \underbrace{\int_{t_0}^{t} e^{-L|t-t_0|} e^{L|s-t_0|} ds}_{\frac{1}{L}(e^{L|t-t_0|} - 1)e^{-L|t-t_0|}}$$

$$= (1 - e^{-Lc}) \|y - \tilde{y}\|_1$$

$\Rightarrow$ Banach FP $\Rightarrow$ unique solution y & $y_{n+1} = T(y_n)$ converges to y.