4 Numerical Methods

To solve nonlinear problems numerically we need to solve linearised versions of the method. We will look at possible linearisations for specific problems and finite element method to solve

4.1 Fixed-point iterative method

Consider Hilbert space X with inner product $(\cdot,\cdot)_X$ inducing norm $\|\cdot\|_X$. where $\langle u,v\rangle = (u,v)_X \; \forall u,v \in X$

Consider the operator $A: X \to X^*$ monotone and Lipschitz continuous with constants M and L respectively. We want to find $u \in X$ such that

$$Au = f$$

for given right hand side $f \in X^*$.

From Theorem 2.11 we have that a unique solution $u \in X$ exists. From the proof of this Theorem, and Banach's fixed point theorem we note that the iteration

$$u_{n+1} = u_n - \varepsilon (Au_n - f)$$

converges to $u \in X$. In the proof we required that

$$k^2 = \underbrace{1 + \varepsilon^2 L^2 - 2\varepsilon M}_{\varphi(\varepsilon)} < 1$$

where $\varepsilon > 0$ is a constant.

The optimal selection of ε is such that $\varphi(\varepsilon)$ is minimised (smallest contraction factor).

So consider $\varphi'(\varepsilon)=0$

$$\Rightarrow 2\varepsilon L^2 - 2M = 0 \Rightarrow \varepsilon = \frac{M}{L^2}$$

Therefore $k^2 = \varphi(\varepsilon) = 1 - \frac{M^2}{L^2}$

Let $X_M \subset X$ be a finite dimensional subspace of X with $\dim(X_M) < \infty$. The goal is to approximate $Au=f$ in X_m.

So, approximate $Au=f$ in X_m: Find $u_m \in X_m$ such that $A_m u_m = f_m$ (Galerkin approximation) where f_m projection of f, i.e. $f_m = P_m^* f$; hence,

$$\langle f_m, z\rangle = (P_m^* f, z\rangle = \langle f, P_m z\rangle = \langle f, z\rangle \quad \forall z \in X_m$$

Notice, that for $z \in X_m$ that

$$\langle Au_m, z\rangle = \langle A_m u_m, z\rangle = \langle f_m, z\rangle = (f, z) \quad \forall z \in X_m$$

and $(Au, z\rangle = \langle f, z\rangle \quad \forall z \in X \supset X_m$

$$\Rightarrow \langle Au_m - Au, z\rangle = 0 \quad \forall z \in X_m$$

We can show that A_m is strongly monotone and Lipschitz continuous with the same constants as for A.

<u>Theorem 4.1</u> The equation $A_m u = f$ has a unique solution $u \in X_m$ and the fixed point iteration
$$u_m^{(n+1)} = u_m^{(n)} - \frac{M}{L^2}\left(A_m u_m^{(n)} - f_m\right)$$
converges to u_m with contraction factor
$$k = \left(1 - \frac{M^2}{L^2}\right)^{1/2}.$$

Typically $u \neq u_m$, so what is the error commited by considering a finite dimensional subspace solution. Define $e_m^{(n)} = u - u_m^{(n)}$, then

$$\|e_m^{(n)}\|_X = \underbrace{\|u - u_m\|_X}_{\mathrm{I}} + \underbrace{\|u_m - u_m^{(n)}\|_X}_{\mathrm{II}}$$

$$\mathrm{I}: \; M\|u-u_m\|_X^2 \le \langle Au - Au_m, u-u_m\rangle$$
$$= \langle Au-Au_m, u-z\rangle \quad \forall z \in X_m$$
$$\le \|Au-Au_m\|_{X^*}\|u-z\|_X \quad \forall z \in X_m$$
$$\le L\|u-u_m\|_X\|u-z\|_X \quad \forall z\in X_m$$
$$\Rightarrow \|u-u_m\|_X \le \frac{L}{M}\|u-z\|_X \quad \forall z \in X_m$$

As X_m is closed

$$\mathrm{I} = \|u-u_m\|_X \le \frac{L}{M}\underbrace{\inf_{v\in X_m}\|u-v\|_X}_{\text{Best approximation in } X_m}$$

Furthermore,

$$\mathrm{II} \le \frac{k^n}{1-k}\|u_m^{(1)}-u_m^{(0)}\|_X \quad \text{from Banach's F.P Theorem}$$

Note that

$$\frac{k^n}{1-k} = \frac{k^n\overbrace{(1+k)}^{\le 2}}{1-k^2} \le \frac{2k^n}{\left(\frac{M}{L}\right)^2}$$

<u>Theorem 4.2 (a priori)</u>

$$\|e_m^{(n)}\|_X \le \frac{L}{M}\inf_{v\in X}\|u-v\|_X + \frac{2L^2}{M^2}\left(1-\frac{M^2}{L^2}\right)^{n/2}\|u_m^{(0)}-u_m^{(1)}\|_X$$

<u>Practical Implementation</u>

Choose basis $\{\phi_1,\dots,\phi_m\} \subset X_m$ and define

$$u_m = \sum_{j=1}^m \alpha_j\phi_j$$

for some unknown vector $\alpha = (\alpha_1,\dots,\alpha_m)^T \in \mathbb{R}^m$

$$\langle A_m u_m, \phi_i\rangle = \langle f_m, \phi_i\rangle \quad i=1,\dots,m$$
$$\langle Au_m, \phi_i\rangle = \langle f, \phi_i\rangle$$

$$\Rightarrow \quad \underbrace{\left\langle A\left(\sum_{j=1}^{m} \alpha_j \phi_j\right), \phi_i \right\rangle}_{F(\alpha)_i} = \underbrace{\langle f, \phi_i \rangle}_{l_i} \quad i = 1, \dots, m$$

Here, we have $F : \mathbb{R}^m \to \mathbb{R}^m$ defined as

$$F(\alpha) = \left(\left\langle A\left(\sum_{j=1}^{m} \alpha_j \phi_j\right), \phi_i \right\rangle \right)_{i=1}^{m}$$

and $l = (l_1, \dots, l_m)^T$. This results in the algebraic (nonlinear) system $F(\alpha) = l$.

Consider solution by fixed point iteration

$$u_m^{(n+1)} = u_m^{(n)} - \frac{M}{L^2}\left(A_m u_m^{(n)} - f_m\right)$$

$$\Rightarrow \left(u_m^{(n+1)}, \phi_i\right)_X = \left(u_m^{(n)}, \phi_i\right)_X - \frac{M}{L^2} \left\langle A_m u_m^{(n)} - f_m, \phi_i \right\rangle$$

for $i = 1, \dots, m$

Let $\alpha^{(n+1)}$ and $\alpha^{(n)}$ be the vectors corresponding to $u_m^{(n+1)}$ and $u_m^{(n)}$; then,

$$\sum_{j=1}^{m} \alpha_j^{(n+1)} (\phi_j, \phi_i)_X = \sum_{j=1}^{m} \alpha_j^{(n)} (\phi_i, \phi_j)_X - \frac{M}{L^2}\left(F(\alpha^{(n)})_i - l_i\right)$$

for $i = 1, \dots, m$

Define the mass matrix $\mathbb{M} \in \mathbb{R}^{m \times m}$ as

$$\mathbb{M}_{ij} = (\phi_j, \phi_i)_X$$

$$\Rightarrow \mathbb{M}\alpha^{(n+1)} = \mathbb{M}\alpha^{(n)} - \frac{M}{L^2}\left(F(\alpha^{(n)}) - l\right), \quad n \geq 0$$

<u>Remark</u> $\mathbb{M}$ is symmetric positive definite.

Now lets consider the boundary value problem in $\Omega \subset \mathbb{R}^2$

$$-\nabla\cdot(\mu(x,|\nabla u|)\nabla u) = 0 \text{ in } \Omega$$
$$u = 0 \text{ on } \partial\Omega$$

where $\mu \in C^0(\bar{\Omega}\times[0,\infty))$ and there exists positive constants $\alpha_1 \geq \alpha_2 > 0$ such that, for $t \geq s \geq 0$ and $x \in \bar{\Omega}$

$$\alpha_2(t-s) \leq \mu(x,t)t - \mu(x,s)s \leq \alpha_1(t-s)$$

<u>Lemma 4.3</u> For vector-valued functions $\underline{u},\underline{v}$

$$|\mu(x,|\underline{u}|)\underline{u} - \mu(x,|\underline{v}|)\underline{v}| \leq \alpha_1|\underline{u}-\underline{v}|$$
$$\alpha_2|\underline{u}-\underline{v}|^2 \leq (\mu(x,|\underline{u}|)\underline{u} - \mu(x,|\underline{v}|)\underline{v})\cdot(\underline{u}-\underline{v})$$

<u>Proof</u> Liu & Barett, M^2AN: 28(6): 725–744, 1994.

Define the weak formulation: Find $u \in X = H_0^1(\Omega)$

(1) $\langle Au, v\rangle = \int_\Omega \mu(|\nabla u|)\nabla u\cdot\nabla v\,dx = 0 \quad \forall v \in H_0^1(\Omega)$

Consider $\|\cdot\|_{1,2}$ which is a full norm in $H_0^1(\Omega)$.

<u>Lemma 4.4</u> The operator A is strongly monotone and Lipschitz continuous with $M = \alpha_2$ & $L = \alpha_1$.

<u>Proof</u>
$$\langle Au - Av, u-v\rangle = \int_\Omega (\mu(|\nabla u|)\nabla u - \mu(|\nabla v|)\nabla v)\cdot\nabla(u-v)\,dx$$
$$\geq \int_\Omega \alpha_2|\nabla(u-v)|^2\,dx$$
$$= \alpha_2|u-v|_{1,2}^2$$

$$|\langle Au - Av, w\rangle| \leq \int_\Omega |\mu(|\nabla u|)\nabla u - \mu(|\nabla v|)\nabla v||\nabla w|\,dx$$
$$\leq \int_\Omega \alpha_1|\nabla(u-v)||\nabla w|\,dx$$
$$\leq \alpha_1\,|u-v|_{1,2}|w|_{1,2}$$

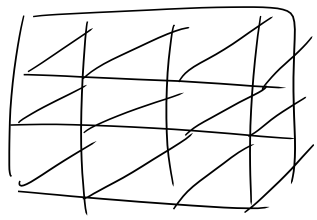
$$\|Au - Av\|_{1,2,*} = \sup_{w \in W_0^{1,2}(\Omega)} \frac{|\langle Au - Av, w\rangle|}{|w|_{1,2}}$$
$$\leq \alpha_1 |u - v|_{1,2} \qquad \square$$

Theorem 4.5 (1) has a unique solution $u \in H_0^1(\Omega)$

Proof Banach F.P Theorem.

Now consider a finite element discretisation

- Partition Ω into triangles T such that $\Omega = \sum_{T \in \mathcal{T}_h} T$ to create triangulation $\mathcal{T}_h$:



(+ certain assumptions on triangulation)

- Denote by h_T diameter of T & let $h = \max_{T \in \mathcal{T}_h} h_T$.

On each element approximate by polynomials of total order $p \geq 1$ and impose continuity across element boundaries.

So we can define the space
$$V_h = \{v \in H_0^1(\Omega) : v|_T \in P_p(K), \forall T \in \mathcal{T}_h\} \subset H_0^1(\Omega).$$

V_h is a finite dimensional subspace of $H_0^1(\Omega)$

So, we now define an iterative Galerkin FEM:

Given initial guess $u_h^{(0)} \in V_h$ we iterate for $n = 0, 1, 2, \ldots$ and find $u_h^{(n+1)} \in V_h$ such that

(2) $\langle u_h^{(n+1)}, v_h\rangle = \langle u_h^{(n)}, v_h\rangle - \frac{M}{L^2}\langle Au_h^{(n)}, v_h\rangle \quad \forall v_h \in V_h.$

All the previous theorems hold; therefore, we have a method that converges to the numerical solution $u_h \in V_h$ with contraction factor

$$k = \left(1 - \frac{M^2}{L^2}\right)^{1/2} < 1.$$

Theorem 4.6 Let $u \in H^{s+1}(\Omega) \cap H^1_0(\Omega)$, with $s \geq 1$, be the weak solution given by (1), $u_h^{(0)} \in V_h$ be any initial guess, and $u_h^{(n)} \in V_h$ the numerical solution after n steps of the fixed point iteration (2); then, for $n \geq 1$, the error bound

$$\|u - u_h^{(n)}\|_{1,2} \leq \frac{CL}{M} h^{\min(p,s)} \|u\|_{s+1,2} + \frac{2L^2}{M^2}\left(1 - \frac{M^2}{L^2}\right)^{\frac{n}{2}} \|u_h^{(0)} - u_h^{(1)}\|_{1,2}$$

holds, where $C > 0$ is independent of h, p, α_1 & α_2.

Proof Follows from Theorem 4.2 and standard finite element approximation results.