3.9 Existence Theory

Let $(\mathcal{A}u)(x) = \sum_{|\alpha| \le k} D^\alpha a_\alpha(x, \delta_k u(x))$,

$p \in (1, \infty)$ and $a_\alpha \in CAR^*(p)$ for $|\alpha| \le k$.

Let V be such that

$$W_0^{k,p}(\Omega) \subset V \subset W^{k,p}(\Omega),$$

Q Banach space of functions on Ω, with norm $\|\cdot\|_Q$ where $C^\infty(\Omega)$ is dense in Q and V continuosly embedded in Q ($V \hookrightarrow Q$). Finally, we have

a) function $\varphi \in W^{k,p}(\Omega)$

b) functional $g \in V^*$ such that for all $v \in W_0^{k,p}(\Omega)$

$$\langle g, v \rangle_V = 0$$

c) functional $f \in Q^*$

We define $A : W^{k,p}(\Omega) \to (W^{k,p}(\Omega))^*$ such that for all $u, v \in W^{k,p}(\Omega)$

$$\langle Au, v \rangle = \sum_{|\alpha| \le k} \int_\Omega a_\alpha(x, \delta_k u(x)) D^\alpha v(x)\, dx.$$

By Theorem 3.12 A bounded and continuous for $W^{k,p}(\Omega) \to (W^{k,p}(\Omega))^*$.

Define operator T on V such that for $u \in V$ Tu is an element from V^* which holds

$$\langle Tu, v \rangle = \langle A(u+\varphi), v \rangle - \langle f, v \rangle_Q - \langle g, v \rangle_V$$

for all $v \in V$.

From concept of weak solution of BVP $(\mathcal{A}, V, Q)$ the set of all weak solutions is identical to the set of solutions of $Tu = 0$ in V. Existence of solutions of this equation can be shown by Theorems 2.17 (Minty-Browder), 2.18 (Leray-Lions), or 2.27 (Existence of monotone, coercive, potential operator). So we need to verify assumptions of these theorems and find assumptions required for $a_\alpha(x, \xi)$ to satisfy these conditions.

<u>Lemma 3.13</u> The space V defined above is reflexive.

<u>Proof</u> $W^{k,p}(\Omega)$ and $W_0^{k,p}(\Omega)$ are reflexive for $p \in (1, \infty)$. Any closed subspace of a reflexive space is reflexive $\Rightarrow$ V reflexive.

<u>Lemma 3.14</u> The operator T defined above is bounded and demicontinuous.

<u>Lemma 3.15</u> If for all $\xi, \eta \in \mathbb{R}^k$, and almost all $x \in \Omega$

$$\sum_{|\alpha| \leq k} \left[a_\alpha(x, \xi) - a_\alpha(x, \eta)\right](\xi_\alpha - \eta_\alpha) \geq 0;$$

then, the operator T defined above is monotone.

<u>Corollary 3.16</u> If equality of the condition in Lemma 3.15 only holds when $\xi = \eta$ then T is strictly monotone.

Lemma 3.17 If there exists constants $c_1, c_2 > 0$, $c_3 \geq 0$ (in case that $V = W_0^{k,p}(\Omega)$, $c_2 \geq 0$) such that $\xi \in \mathbb{R}^K$ and almost all $x \in \Omega$

$$\sum_{|\alpha| \leq k} \xi_\alpha a_\alpha(x, \xi) \geq c_1 \sum_{|\alpha|=k} |\xi_\alpha|^p + c_2 |\xi_{\underline{0}}|^p - c_3$$

where $\underline{0} = (0, \dots, 0)$; then, T is coercive.

Proof By definition of T and construction of V

$$\lim_{\|u\|_{k,p} \to \infty} \frac{\langle Tu, u \rangle}{\|u\|_{k,p}} = \infty$$

$$\Longleftrightarrow \lim_{\|u\|_{k,p} \to \infty} \frac{\langle A(u+\varphi), u \rangle}{\|u\|_{k,p}} = \infty \qquad (*)$$

$$\Longleftrightarrow \lim_{\|u\|_{k,p} \to \infty} \frac{1}{\|u\|_{k,p}} \int_\Omega \sum_{|\alpha| \leq k} a_\alpha(x, \delta_k u(x) + \delta_k \varphi(x)) D^\alpha u(x)\, dx = \infty$$

We show $(*)$ only in case that $a_\alpha \in CAR(p)$; the general case is analogous. From conditions on a_α in the statement and equivalence of the standard $\|\cdot\|_{k,p}$ norm and $|||u||| = \|u\|_p + |u|_{k,p}$; then,

$$\langle A(u+\varphi), u+\varphi \rangle \geq \int_\Omega \Big[c_1 \sum_{|\alpha|=k} |D^\alpha u + D^\alpha \varphi|^p + c_2 |u+\varphi|^p - c_3 \Big] dx$$

$$\geq c_4 \|u+\varphi\|_{k,p}^p - c_3 |\Omega|$$

$$\langle A(u+\varphi),\varphi\rangle \leq \sum_{|\alpha|\leq k} \|a_\alpha(x,\delta_k u+\delta_k\varphi)\|_q \|D^\alpha\varphi\|_p$$
$$\leq \Big(\sum_{|\alpha|\leq k}\|a_\alpha(x,\delta_k u+\delta_k\varphi)\|_q^q\Big)^{1/q}\|\varphi\|_{k,p}$$
$$\leq \big(\tilde{C}_1+\tilde{C}_2\|u+\varphi\|_{k,p}^p\big)^{1/q}\|\varphi\|_{k,p} \qquad \text{(Th 3.6) (bound on Nemyckii op.)}$$
$$\leq \big(C_5 + C_6\|u+\varphi\|_{k,p}^{p/q}\big)\|\varphi\|_{k,p}$$
$$= \big(C_5 + C_6\|u+\varphi\|_{k,p}^{p-1}\big)\|\varphi\|_{k,p}$$

Then,
$$\langle A(u+\varphi),u\rangle = \langle A(u+\varphi),u+\varphi\rangle - \langle A(u+\varphi),\varphi\rangle$$
$$\geq C_4\|u+\varphi\|_{k,p}^p - C_6\|\varphi\|_{k,p}\|u+\varphi\|_{k,p}^{p-1}$$
$$- C_5\|\varphi\|_{k,p} - C_3|\Omega|$$

Divide by $\|u\|_{k,p}$ and take limit as $\|u\|_{k,p}\to\infty$. □

Theorem 3.18

Let the coefficients $a_\alpha(x,\xi)$ meet the conditions

1) $a_\alpha \in CAR^k(p)$
2) $\sum_{|\alpha|\leq k}[a_\alpha(x,\xi)-a_\alpha(x,\eta)](\xi_\alpha-\eta_\alpha)\geq 0 \quad \forall \xi,\eta\in\mathbb{R}^K$
3) $\sum_{|\alpha|\leq k}\xi_\alpha a_\alpha(x,\xi)\geq C_1\sum_{|\alpha|\leq k}|\xi_\alpha|^p + C_2|\xi_{(0,\dots,0)}|^p - C_3$

   where $C_1>0$, $C_2>0$, and $C_3\geq 0$ are constants.

Then, there exists at least one weak solution of the boundary value problem (A,V,Q). Furthermore, if equality in 2 holds only when $\xi=\eta$ then the solution is unique.

Proof From Lemmas 3.13, 3.14, 3.15 and 3.17 we can show the conditions of Minty-Browder (Theorem 2.17) are met. Therefore, we can show $Tu=0$ has a solution $\Rightarrow$ existence of weak solution to BVP. Corollary 3.16 then gives uniqueness of the solution. $\square$