Theorem 3.5

Let $p_1, \dots, p_m$ and q be real numbers, with $p_i \geq 1$ for $i = 1, \dots, m$, $q \geq 1$, and $m \in \mathbb{N}$. Let f be function defined for $x \in \Omega$, $\xi \in \mathbb{R}^m$ which satisfies the Caratheodory conditions, and denote by

$$\mathcal{N}(u_1, \dots, u_m)(x) = f(x, u_1(x), \dots, u_m(x)), \quad x \in \Omega$$

the Nemyckii operator for f where $u_i = u_i(x), i = 1, \dots, m$ are functions defined over Ω. Then,

1) For the m-tuple functions $u_i \in L^{p_i}(\Omega), i = 1, \dots, m$

$$\mathcal{N}(u_1, \dots, u_m) \in L^q(\Omega)$$

if the following growth condition is met:
$\exists g \in L^q(\Omega)$ and constant $C \geq 0$ almost everywhere in Ω and such that for all $\xi \in \mathbb{R}^m$

$$|f(x, \xi_1, \dots, \xi_m)| \leq g(x) + C \sum_{i=1}^{m} |\xi_i|^{p_i/q} \qquad (2)$$

2) If the growth condition is met, the $\mathcal{N}$ is a well-defined, continuous, and bounded operator from the Cartesian space $L^{p_1}(\Omega) \times \dots \times L^{p_m}(\Omega)$ to $L^q(\Omega)$.

Theorem 3.6

Let $\mathcal{N} : L^{p_1}(\Omega) \times \dots \times L^{p_m}(\Omega) \to L^q(\Omega)$ be the Nemyckii operator with function $f : \Omega \times \mathbb{R}^n \to \mathbb{R}$ satisfying the Caratheodory conditions and growth condition (2);
Then, $\mathcal{N}$ is continuous and bounded such that

$$\|\mathcal{N} u\|_q \leq C_1 + C_2 \sum_{i=1}^{m} \|u_i\|_{p_i}^{p_i/q}$$

for all $u \in L^{p_1}(\Omega) \times \ldots \times L^{p_m}(\Omega)$, where
$C_1 = (m+1)^{(q-1)/q} \|g\|_q$ and $C_2 = (m+1)^{(q-1)/q} C$.

If $p \geq 1$ the function $h(x, \xi) \in CAR$, $x \in \Omega$, $\xi \in \mathbb{R}^K$, satisfies the growth condition

$$|h(x,\xi)| \leq g(x) + C \sum_{|\beta| \leq k} |\xi_\beta|^{p-1}, \quad \frac{1}{p} + \frac{1}{q} = 1$$

where $g \in L^q(\Omega)$ & $C > 0$ we denote

$$h \in CAR(p).$$

Note this is the same as ② if $p = p_1 = \ldots = p_m$ & $\frac{1}{p} + \frac{1}{q} = 1$

3.5 Sobolev Embeddings

Theorem 3.7 Let Ω subdomain of $\mathbb{R}^n$, $k, n \in \mathbb{N}$, $p \geq 1$.

Then

(i) If $kp < n$ then for arbitrary q such that $1 \leq q \leq \frac{np}{n-kp}$

$$W^{k,p}(\Omega) \hookrightarrow L^q(\Omega)$$

(ii) If $kp = n$ then for arbitrary r such that $1 \leq r < \infty$

$$W^{k,p}(\Omega) \hookrightarrow L^r(\Omega)$$

(iii) If $kp > n$ then

$$W^{k,p}(\Omega) \hookrightarrow C^0(\overline{\Omega})$$

Theorem 3.8 Let Ω be subdomain of $\mathbb{R}^n$, $k, n \in \mathbb{N}$, $p \geq 1$.
Choose multi-index β, $|\beta| \leq k$, and $u \in W^{k,p}(\Omega)$. Then,

(i) if $|\beta| > k - \frac{n}{p}$, then $D^\beta u \in L^{q(\beta)}$, where $q(\beta) = \frac{np}{n-(k-|\beta|)p}$

(ii) if $|\beta| = k - \frac{n}{p}$ then $D^\beta u \in L^{q(\beta)}(\Omega)$, where $q(\beta) \geq 1$ is arbitrary

(iii) if $|\beta| < k - \frac{n}{p}$, then $D^\beta u \in C^0(\overline{\Omega})$.

Additionally, there exists a constant $C > 0$ such that for all $u \in W^{k,p}(\Omega)$

$$\|D^\beta u\|_X \leq C\|u\|_{k,p}$$

where $X = L^{q(\beta)}(\Omega)$ or $C^q(\Omega)$ dependent on (i), (ii) or (iii).

3.6 Weak solution of differential equations

Let $p>1$ and $k \in \mathbb{N}$. Suppose that for the coefficients in the divergence form ①

$$a_\alpha \in CAR(p) \quad \text{for} \quad |\alpha| \leq k.$$

Since, for $u \in W^{k,p}(\Omega)$, $\delta_k u \in [L^p(\Omega)]^K$; then, by Theorems 3.4–3.6 the function $a_\alpha(x, \delta_k u(x))$ is continuous for $W^{k,p}(\Omega)$ to $W^{k,q}(\Omega)$, $\frac{1}{p}+\frac{1}{q}=1$, and the estimate

$$\|a_\alpha(x, \delta_k u(x))\|_q \leq C_{\alpha_1} + C_{\alpha_2} \sum_{|\beta| \leq k} \|D^\beta u\|_p^{p-1},$$

where C_{α_1} and C_{α_2} are positive constants, holds.

Then, from Hölder's inequality for $v \in W^{k,p}(\Omega)$ & $|\alpha| \leq k$

$$\left| \int_\Omega a_\alpha(x, \delta_k u(x)) D^\alpha v(x)\,dx \right| \leq \left(C_{\alpha_1} + C_{\alpha_2} \sum_{|\beta| \leq k} \|D^\beta u\|_p^{p-1} \right) \|D^\alpha v\|_p$$

We define the formal differential operator

$$(\mathcal{A}u)(x) = \sum_{|\alpha| \leq k} D^\alpha a_\alpha(x, \delta_k u(x)) \qquad ③$$

and we have that

$$\langle A u, v \rangle = \sum_{|\alpha| \leq k} \int_\Omega a_\alpha(x, \delta_k u(x)) D^\alpha v(x)\,dx, \quad v \in W^{k,p}(\Omega) \qquad ④$$

where A maps from $W^{k,p}(\Omega)$ to dual $(W^{k,p}(\Omega))^*$.

We call $u \in W^{k,p}(\Omega)$ the weak solution of the formal differential equation $\mathcal{A}u = f$ if for all

$v \in W_0^{k,p}(\Omega)$ $\quad \langle Au, v \rangle = \langle f, v \rangle$

where $f \in (W_0^{k,p}(\Omega))^*$ is a functional defined by

$$\langle f, v \rangle = \int_\Omega f(x) v(x) dx \qquad \forall v \in L^p(\Omega)$$

for the function $f \in L^q(\Omega)$.

3.7 Boundary Value Problem

The solution of a boundary value problem is to find the solution of ① satisfying the boundary conditions. Given ① of order $2k$, and functions $\varphi_0, \dots, \varphi_{k-1}$ defined on $\partial\Omega$ with

$$D^j u\big|_{\partial\Omega} = \frac{\partial^j u}{\partial \underline{n}^j} = \varphi_j \text{ on } \partial\Omega, \; j = 0, \dots, k-1$$

then finding solution is the Dirichlet problem.

Definition 3.9

Let $\mathcal{A}$ be the formal differential operator from ③ of order $2k$, f continuous linear functional over $W_0^{k,p}(\Omega)$ and φ function over $W^{k,p}(\Omega)$; then, $u \in W^{k,p}(\Omega)$ is the weak solution to the Dirichlet problem if

(i) $u - \varphi \in W_0^{k,p}(\Omega)$

(ii) for every $v \in W_0^{k,p}(\Omega)$

$$\sum_{|\alpha| \le k} \int_\Omega a_\alpha(x, \delta_k u(x)) D^\alpha v(x) dx = \langle f, v \rangle$$

holds

Remark Condition (i) says that for $|\beta| \le k-1$ such that $D^\beta(u-\varphi)\big|_{\partial\Omega} = 0$ on $D^\beta u = D^\beta \varphi$ on $\partial\Omega$; therefore, the sought solution u and its derivatives take prescribed values on $\partial\Omega$.

Remark If function values φ are only known on boundary $\partial\Omega$ then finding φ involves extending to the whole domain Ω, such that $\varphi \in W^{k,p}(\Omega)$. Existence of such a solution is non-trivial.

Definition 3.10

Let $\mathcal{A}$ be the differential operator from (3) with coefficients $a_\alpha \in CAR(p)$, for $|\alpha| \leq k$, $p > 1$, and $\mathcal{V}$ a set of linear functions defined on Ω such that

$$C_0^\infty(\Omega) \subset \mathcal{V} \subset C^\infty(\bar{\Omega})$$

Denote by V the closure of $\mathcal{V}$ in the $\|\cdot\|_{k,p}$; then,

$$W_0^{k,p}(\Omega) \subset V \subset W^{k,p}(\Omega).$$

Furthermore, let Q be a Banach space on Ω with norm $\|\cdot\|_Q$ such that $C_0^\infty(\Omega)$ is dense in Q and V is embedded in Q $(V \hookrightarrow Q)$. Finally, define

(a) function $\varphi \in W^{k,p}(\Omega)$

(b) functional $g \in V^*$ such that for all $v \in W_0^{k,p}(\Omega)$

$$\langle g, v \rangle_V = 0$$

holds, and

(c) functional $f \in Q^*$.

Then, we can say $u \in W_0^{k,p}(\Omega)$ is a weak solution of the boundary value problem $(\mathcal{A}, V, Q)$ if

(i) $u - \varphi \in V$ and

(ii) for all $v \in V$

$$\sum_{|\alpha| \leq k} \int_\Omega a_\alpha(x, \delta_k u(x)) D^\alpha v(x)\, dx = \langle f, v \rangle_Q + \langle g, v \rangle_V.$$

<u>Remark</u> Looking for weak solutions of the boundary value problem $(\mathcal{A}, V, Q)$ means solving the operator equation $\mathcal{A}u = \Phi$ on the set $\{u \in W^{k,p}(\Omega) : u - \varphi \in V\}$. Here $\mathcal{A}$ is operator defined by (3) and $\Phi \in V^*$ a functional defined by

$$\langle \Phi, v \rangle = \langle f, v \rangle_Q + \langle g, v \rangle_V .$$

3.8 Generalisation of growth condition

<u>Theorem 3.11</u> Let $k \in \mathbb{N}, p \geq 1, r \geq 1$, $f(x, \xi) \in CAR$ be a function defined on $x \in \Omega$ and $\xi \in \mathbb{R}^K$. Suppose there exists a continuous function $c = c(t) \geq 0$, defined for $t \geq 0$, and function $g \in L^r(\Omega)$ such that for all $\xi \in \mathbb{R}^K$ and for almost all $x \in \Omega$ it holds that

$$|f(x,\xi)| \leq c\Big(\sum_{|\beta| \leq k - \frac{n}{p}} |\xi_\beta|\Big)\Big[g(x) + \sum_{k-\frac{n}{p} \leq |\beta| \leq k} |\xi_\beta|^{\frac{q(\beta)}{r}}\Big]$$

where $q(\beta) = \dfrac{np}{n - (k - |\beta|)p}$ if $|\beta| > k - \dfrac{n}{p}$, or

$q(\beta) \geq 1$ arbitrary if $|\beta| = k - \dfrac{n}{p}$.

Then, for each $u \in W^{k,p}(\Omega)$, $f(x, \delta_k u(x)) \in L^r(\Omega)$, and the Nemyckii operator defined by $f(\mathcal{N}u = f(x, u(x)), x \in \Omega)$ is continuous and bounded from $W^{k,p}(\Omega)$ to $L^r(\Omega)$.

<u>Proof</u> Boundedness is trivial. Continuity more difficult.

<u>Remark</u> If $kp \leq n$ then there does not exist a multi-index β such that $|\beta| \leq k - \frac{n}{p}$. In this case the condition above is modified to a constant c rather than a function c.

Theorem 3.12 Let $\mathcal{A}$ be a formal differential operator of order $2k$

$$(\mathcal{A}u)(x) = \sum_{|\alpha| \le k} D^\alpha a_\alpha(x, \delta_k u(x))$$

and let the functions $a_\alpha \in CAR$ satisfy, for almost all $x \in \Omega$ and all $\xi \in \mathbb{R}^k$ the growth condition

$$|a_\alpha(x,\xi)| \le C_\alpha\Big(\sum_{|\beta|<k-\frac{n}{p}} |\xi_\beta|\Big)\Big(g_\alpha(x) + \sum_{k-\frac{n}{p} \le |\beta| \le k} |\xi_\beta|^{r(\alpha,\beta)}\Big)$$

where $p>1$ and

(i) $C_\alpha(t)$ is a non-negative continuous function of variable $t \ge 0$ with $C_\alpha = \text{constant}$ for $k-\frac{n}{p}<0$,

(ii) $g_\alpha \in L^s(\Omega)$ where

$$s = \begin{cases} \frac{q(\alpha)}{q(\alpha)-1} & \text{if } |\alpha| \ge k-\frac{n}{p} \\ 1 & \text{if } |\alpha| < k-\frac{n}{p} \end{cases}$$

(iii)

$$r(\alpha,\beta) = \begin{cases} \frac{(q(\alpha)-1)q(\beta)}{q(\alpha)} & \text{if } |\alpha| \ge k-\frac{n}{p} \\ q(\beta) & \text{if } |\alpha| < k-\frac{n}{p} \end{cases}$$

(iv)

$$q(\gamma) = \begin{cases} \frac{np}{n-(k-|\gamma|)p} & \text{if } |\gamma| > k-\frac{n}{p} \\ \ge 1 \text{ arbitrary} & \text{if } |\gamma| = k-\frac{n}{p} \end{cases}$$

Then, the operator A defined by the relation

$$\langle Au, v\rangle = \sum_{|\alpha| \le k} \int_\Omega a_\alpha(x, \delta_k u(x)) D^\alpha v(x)\, dx \quad \forall v \in W^{k,p}(\Omega)$$

is bounded and continuous from $W^{k,p}(\Omega)$ to $(W^{k,p}(\Omega))^*$.

We denote a function $a_\alpha \in CAR$ which meets (+) by

$$a_\alpha \in CAR^+(p).$$