2.4 Potential Operators

Definition 2.19 Let X be a real normed space. We say that operator $A: X \to X^*$ is a potential operator if there exists a functional F on X such that at each $x \in X$ there exists a Gâteaux derivative $\text{grad}\, F(x) \equiv F'(x): X \to X^*$ such that $A = \text{grad}\, F$; i.e.,

$$\langle Ax, y \rangle = \lim_{t \to 0} \frac{1}{t} \left(F(x+ty) - F(x) \right) \quad \forall x, y \in X.$$

Functional F is called the potential of A.

Remark This definition can be extended to $T: X \to Y$, where Y is linearly isometric isomorphic to X^* in a similar manner to how we extend monotonicity.

Lemma 2.20 $A: X \to X^*$ is radially continuous potential operator with potential F. Then, for any $x \in X$

$$F(x) = F(0) + \int_0^1 \langle Atx, x \rangle dt$$

Proof Choose $x \in X$ and define $\varphi(t) = F(tx)$ for $t \in [0,1]$

$$\varphi'(t) = \lim_{s \to 0} \frac{1}{s} \left(F(tx + sx) - F(tx) \right) = \langle Atx, x \rangle$$

A radially continuous $\Rightarrow$ $\langle Atx, x \rangle = \varphi'(t)$ continuous on $[0,1]$

$$\Rightarrow F(x) - F(0) = \varphi(1) - \varphi(0) = \int_0^1 \varphi'(t) dt = \int_0^1 \langle Atx, x \rangle dt$$

Lemma 2.21 X reflexive Banach space, $A: X \to X^*$ demicontinuous. Then, following statements are equivalent:

a) A is a potential operator

b) For any $x, y \in X$

$$\int_0^1 \langle Atx, x \rangle dt - \int_0^1 \langle Aty, y \rangle dt = \int_0^1 \langle A(y + t(x-y)), x - y \rangle dt$$

c) For any $x,y \in X$ and continuous differentiable representation $u:[0,1] \to X$ such that $u(0)=x$ & $u(1)=y$ holds
$$\int_0^1 \langle Atx, x\rangle dt - \int_0^1 \langle Aty, y\rangle dt = \int_0^1 \langle Au(t), u'(t)\rangle dt$$

Remark If assumptions of Lemma 2.21 are met then $A: X \to X^*$ potential operator if and only if for every continuous differentiable function $u:[0,1] \to X$ such that $u(0)=u(1)$ it holds that
$$\int_0^1 \langle Au(\tau), u'(\tau)\rangle d\tau = 0$$

Remark Lemma 2.21 a, b holds if A radially continuous

Lemma 2.22 X reflexive Banach space and $A: X \to X^*$ has at every $x \in X$ a Gâteaux derivative $A'(x)$ such that $\forall x,y,h$ the function
$$\varphi(s,t) = \langle A'(h+sx+ty)x, y\rangle$$
is continuous on $[0,1]$. Then, the following are equivalent

a) A is a potential operator

b) $\forall x,y,h \in X \quad \langle A'(h)x, y\rangle = \langle A'(h)y, x\rangle$

Theorem 2.23 X real normed linear vector space and f smooth functional (at each point $x \in X$ there exists Gâteaux derivative $f'(x) \in X^*$ of f). Then, the following are equivalent

a) f is convex

b) $A \equiv f': X \to X^*$ is monotone; i.e.,
$$\langle Ax - Ay, x-y\rangle = \langle f'(x) - f'(y), x-y\rangle \geq 0 \quad \forall x,y \in X$$

c) For any $x,y \in X \quad f(y) \geq f(x) + \langle Ax, y-x\rangle$

d) For any $x,y \in X \quad f(x+sy) \equiv \psi(s)$ is convex

Lemma 2.24 $A: X \to X^*$ potential operator with potential F. Then, for $u \in X$ to be a solution of $Au = f$, $f \in X^*$, it is sufficient for the following condition to be fulfilled

$$F(u) - \langle f, u \rangle = \min_{v \in X} \left(F(v) - \langle f, v \rangle \right)$$

If A is monotone, then this condition is necessary.

Proof

$F(u) - \langle f, u \rangle = \min_{v \in X} (F(v) - \langle f, v \rangle)$; i.e., the functional f defined on X $g(v) = F(v) - \langle f, v \rangle$, $v \in X$, acquires its minimum at u and has at every $v \in X$ a Gâteaux derivative $g'(v)$:

$$\langle g'(v), h \rangle = \langle \operatorname{grad} F(v), h \rangle - \langle f, h \rangle \iff g'(v) = F'(v) - f$$

Then, $g'(v) = 0$, and therefore for any $h \in X$

$$0 = \langle \operatorname{grad} F(u), h \rangle - \langle f, h \rangle = \langle Au - f, h \rangle$$

and, thus, $Au = f$.

Suppose, additionally, A monotone and $Au = f$.

Then, for any $v \in X$ by Th 2.23 (c)

$$F(v) \geq F(u) + \langle Au, v - u \rangle;$$

hence, $(F(v) - \langle f, v \rangle) - (F(u) - \langle f, v \rangle) = F(v) - F(u) - \langle Au, v - u \rangle \geq 0$

$\Rightarrow u$ minimum.

Lemma 2.25 Every monotone potential operator is demicontinuous.

Lemma 2.26 $A: X \to X^*$ monotone potential operator. Then, for $u \in X$ be solution of $Au = f$, $f \in X^*$ it is necessary and sufficient for

$$\int_0^1 \langle Atu, u \rangle dt - \langle f, u \rangle = \min_{v \in X} \left[\int_0^1 \langle Atv, v \rangle dt - \langle f, v \rangle \right]$$

Theorem 2.27 $A: X \to X^*$ monotone, coercive, and potential operator. Then, $Au = f$ has a solution for every right hand side $f \in X^*$. If A is strictly monotone then the solution is unique.

Corollary 2.28 Potential $F(x) = \int_0^1 \langle Atx, x\rangle dt$, $x \in X$, of monotone, coercive, potential operator $A: X \to X$ is bounded from below.

Proof $u \in X$ solution to $Au = 0$; then, $F(u) \le F(v)$ $\forall v \in X$ (Lemma 2.24)

2.5 Dual Functional

F functional defined on reflexive Banach space. If $F(x) = +\infty$ $\forall x \in X$ then F is trivial; otherwise, it is non-trivial. For non-trivial F, F^* defined as X^* as

$$F^*(x^*) = \sup_{x \in X} (\langle x^*, x\rangle - F(x)), \quad x^* \in X^*$$

is called the dual functional (associated, adjoint) to F

Lemma 2.29 F non-trivial functional. Then,

1) For each $x \in X$ and $x^* \in X^*$
$$F(x) + F^*(x^*) \ge \langle x^*, x\rangle.$$
2) F^* convex and weakly lower semi-continuous
3) If F weakly lower semicontinuous & convex
$$\Rightarrow F^* \text{ is non-trivial}$$

For non-trivial $F^*: X^* \to (-\infty, +\infty)$; then, F^{**} defined by

$$F^{**}(x) = \sup_{x \in X^*} (\langle x^*, x\rangle - F^*(x^*))$$

Theorem 2.30 F non-trivial functional; then, following equivalent:

1) F weakly lower semi-continuous & convex
2) $F = F^{**}$

Theorem 2.31 F finite, weakly lower semicontinuous, convex, and $A: X \to X^*$. Assume $\exists$ inverse $A^{-1}: X^* \to X$; then, following equivalent

1) F potential for $A: X \to X^*$; i.e. $\operatorname{grad} F = A$
2) A radially continuous and for any $x, y \in X$
$$F(x) + F^*(Ax) = \langle Ax, x\rangle, \quad F(y) \geq F(x) + \langle Ax, y-x\rangle$$
3) A^{-1} radially continuous and for any $x^*, y^* \in X^*$
$$F(A^{-1}x^*) + F^*(x^*) = \langle x^*, A^{-1}x^*\rangle$$
$$F(y^*) \geq F(x^*) + \langle y^* - x^*, A^{-1}x^*\rangle$$
4) F^* potential of $A^{-1}: X^* \to X$; i.e, $\operatorname{grad} A^{**} = A^{-1}$.

Corollary 2.32 $A: X \to X^*$ and $\exists$ inverse $A^{-1}: X^* \to X$; then, A monotone potential operator $\Leftrightarrow A^{-1}$ monotone, potential.

Theorem 2.33 $A: X \to X^*$ strictly monotone, coercive, potential operator. Then, the exists inverse $A^{-1}: X^* \to X$ which is strictly monotone potential operator.

The functional
$$F(x) = \int_0^1 \langle Atx, x\rangle dt, \quad x \in X$$
is the potential of A and for any $x \in X$ & $x^* \in X^*$
$$F^*(x^*) = F^*(0) + \int_0^1 \langle x^*, A^{-1}tx^*\rangle dt$$
$$F^*(0) = -F(A^{-1}0)$$
$$F(x) + F^*(x^*) - \langle x^*, x\rangle \geq 0$$
$$F(x) + F^*(x^*) - \langle Ax, x\rangle = 0$$
where F^* is the potential of A^{-1}.

Corollary 2.34 $A: X \to X^*$ strictly monotone, coercive, potential operator with potential F. For any $f \in X^*$ $\exists!$ solution $u \in X$ of $Au = f$ which minimises the potential of the problem $G = F - f$ and

$$G(u) \equiv F(u) - \langle f, u \rangle = \min_{v \in X} \left[\int_0^1 \langle Atv, v \rangle dt - \langle f, v \rangle \right]$$

$$= -\int_0^1 \langle f, A^{-1}(tf) \rangle dt + \int_0^1 \langle AtA^{-1}0, A^{-1}0 \rangle dt.$$

2.6 Ritz Method

From theory of linear operator equations it is known that Galerkin & Ritz methods for solving equations with symmetric, non-negative operators give same approximation under certain conditions.

Definition 2.35 $A: X \to X^*$ potential operator with potential F, $\{h_i\}$ be dense set in X; then u_n is called the nth Ritz approximation of $Au = f$, $f \in X^*$, if it holds

$$F(u_n) - \langle f, u_n \rangle = \min_{v \in X_n} (F(v) - \langle f, v \rangle)$$

where $X_n = \operatorname{span}\{h_i, i = 1, \dots, n\}$.

Theorem 2.36 $A: X \to X^*$ monotone, potential operator with potential F. Then, u_n is Ritz approximation of $Au = f$, $f \in X^*$, if and only if u_n is the Galerkin approximation in X_n; i.e., $A_n u_n = f_n$, $n \in \mathbb{N}$, where f_n projection of f onto X_n

Proof

- u_n Ritz approximation. Then, for any $t \in \mathbb{R}$ and $h \in X_n$, set $v = u_n + th$,

$$F(u_n + th) - F(u_n) \geq t\langle f, h \rangle.$$

Hence, for any $h \in X_n$

$$0 = \lim_{t \to 0} \frac{1}{t}\left(F(u_n+th) - F(u_n)\right) - \langle f, h\rangle = \langle Au_n - f, h\rangle$$

$$\Rightarrow \langle Au_n, h\rangle = \langle f, h\rangle \quad \forall h \in X_n$$

and, thus, u_n Galerkin approximation in X_n.

- $u_n \in X_n$ Galerkin approximation. Then, by Th. 2.31 for any $x, y \in X$ $F(y) \geq F(x) + \langle Ax, y - x\rangle$

Hence, for any $v \in X_n$ $(y = v,\ x = u_n)$

$$\left(F(v) - \langle f, v\rangle\right) - \left(F(u_n) - \langle f, u_n\rangle\right) \geq \langle Au_n - f, v - u_n\rangle = 0;$$

Thus, u_n Ritz approximation. $\square$

Theorem 2.37 $A: X \to X^*$ strictly monotone, coercive, potential operator which satisfies (S). Then, $Au = f$ has a unique solution for each right hand side $f \in X$ to which the Ritz approximations converges.