2.3 Existence Theory

Consider existence of a solution to $Au=f$, where $A: X \to X^*$, $f \in X$ and X is a reflexive Banach space.

Theorem 2.11 Let $A: X \to X$, X Hilbert space, be strongly monotone & Lipschitz continuous. Then, for each $f \in X$ the equation $Au=f$ has a unique solution.

Proof – See practical 3 (Q1(b))

Theorem 2.12 Let H be a Hilbert space and the operator $A: H \to H$ meet the following conditions:

1) A is monotone
2) A is continuous for every finite dimensional subspace $M \subset H$
3) A is coercive; i.e, $\lim_{\|x\| \to \infty} \frac{(Ax, x)}{\|x\|} = \infty$

Then, the operator A is surjective; i.e, the equation $Ax=y$ has a solution for every right hand side $y \in H$. Moreover, if A is strictly monotone the solution is unique.

Proof – See exercise

Theorem 2.13 Let H be a Hilbert space and $A: H \to H$ meet conditions:

1) A is monotone
2) A is continuous for every finite dimensional subspace $M \subset H$
3) A is weakly coercive, i.e $\lim_{\|x\| \to \infty} \|Ax\| = \infty$

Then, A is surjective. Moreover, if A is strictly monotone the solution is unique.

Now consider solution to $Au=b$ in reflexive Banach space X, where $A: X \to X^*$, $b \in X^*$. For simplicity, we assume that X is separable (not required!)

Definition 2.14

Let X_n be a finite dimensional subspace of a Banach space X. Then, the Galerkin approximation of the equation $Au=b$ on X_n is: Find $u_n \in X_n$ such that

$$\langle Au_n, v\rangle = \langle b, v\rangle \quad \forall v \in X_n.$$

This can be understood as $A_n u_n = b_n$, where $A_n : X_n \to X_n^*$ is the restriction of A to X_n, and $A_n u_n$, b_n are functionals on X restricted to X_n.

Remark Let $X_n = \text{span}\{v_i, i=1,\dots,n\}$ be finite dimensional subspace of Banach space X. Define P_n as projection from X to X_n (cont. linear operator) and construct dual operator for this projection $P_n^* : X_n^* \to X^*$.

$$\langle f_n, P_n x\rangle = \langle P_n^* f_n, x\rangle \quad \forall f_n \in X_n^* \ \& \ x \in X.$$

Let $\quad A_n = P_n^* A P_n$ and $b_n = P_n^* b$.

Then, $A_n u_n = b_n$ is the Galerkin approximation to $Au=b$ on X_n.

By definition of Galerkin approximation the following are equivalent:

- Find $u_n \in X_n$ such that $\langle Au_n, v\rangle = \langle b, v\rangle \quad \forall v \in X_n$
- Find $u_n \in X_n$ such that $\langle P_n^* A P_n u_n, v\rangle = \langle P_n^* b, v\rangle \quad \forall v \in X_n$
- Find $u_n \in X_n$ such that $A_n u_n = b_n$

A_n has has same characteristics as A (monotone, coercive, etc.), due to fact P_n cont. linear operator.

Theorem 2.15 (Basic Theorem of Surjectivity)

Let X be a separable, reflexive Banach space and the operator $A: X \to X^*$ meets the following conditions:

- coercive
- continuous on finite dimensional subspaces
- bounded
- satisfies condition $(M)_0$

Then, the operator A is surjective; i.e, solution of $Au=b$ exists for every right hand side $b \in X^*$. Additionally, A^{-1} is bounded; i.e, $\exists$ function $N: \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\forall u \in X \quad \|u\| \leq N(\|Au\|)$$

Proof: 5 steps:

1) Construct sequence of finite dimensional subspaces $\{X_n\}$ that approximate Banach space X in the limit. Gives sequence of Galerkin approximations to $Au=b$.
2) Prove existence of solution to $A_n u_n = b_n$. Obtain sequence of approximate solutions $\{u_n\}, u_n \in X_n$, of $Au=b$.
3) Prove can select weakly convergent subsequence $u_{n_k} \rightharpoonup u$ so that $Au_{n_k} \rightharpoonup b$.
4) Prove weak limit u solution of $Au=b$.
5) Finally, prove all solutions of $Au=b$ are bounded.

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1) X is separable: $\exists$ countable dense subset from which we can select linearly independent sequence $\{v_n\}$, $\|v_n\|=1$ such that linear span is a dense set in X. Let

$$X_n := \operatorname{span}\{v_i\}_{i=1}^{n} \text{ and } b_n := b|_{X_n^*}.$$

It follows that sequence $\{X_n\}$ is densely bounded in X; i.e., $\forall x \in X \ \exists \{x_n\}, x_n \in X_n$ such that $x_n \to x$ in norm of space X. Since, $X_n \subset X$ then $X \subset X_n^*$, and, therefore, $b \in X_n^*$. Find $u_n \in X_n$ such that $A_n u_n = b_n$

(Galerkin approx)

2) Select $x_n \in X_n^*$, and there exists

$$\alpha = (\alpha_i)_{i=1}^n, \alpha_i \in \mathbb{R} \text{ such that } x_n = \sum_{i=1}^n \alpha_i v_i$$

Construct $g: \mathbb{R}^n \to \mathbb{R}^n$ as

$$g(\alpha) = (g_j)_{j=1}^n, \quad g_j = \langle A x_n, v_j \rangle,$$

From this and assumptions $g: \mathbb{R}^n \to \mathbb{R}^n$ is continuous and coercive. Possible to show that $g: \mathbb{R}^n \to \mathbb{R}^n$ is surjective.

Then, $\quad y = (\langle b, v_j \rangle)_{j=1}^n \in \mathbb{R}^n$

and, hence, $\exists$ solution (α) to $g(\alpha) = y$. Let

$u_n = \sum_{j=1}^n \alpha_j v_j$; then, $A_n u_n = b_n$.

3) From coercivity of A show $\{u_n\}$ bounded. From Galerkin approximation $\langle A u_n, u_n \rangle = \langle b, u_n \rangle$ and, thus,

$$\frac{\langle A u_n, u_n \rangle}{\|u_n\|} \leq \|b\|_{X^*}$$

As A coercive $\exists N: \mathbb{R}_+ \to \mathbb{R}_+$ such that for $r > 0$

$$① \quad \frac{\langle A u, u \rangle}{\|u\|} \geq r \quad \text{for } u \in X, \|u\| \geq N(r)$$

From, these we get

$$\|u_n\| \leq N(\|b\|_{X^*}) \qquad \forall n$$

As X and X^* reflexive, then by Theorem 1.4 there exists a subsequence that weakly converges to $u \in X$ and from this a subsequence $\{u_{n_k}\}$ can be selected such that $\{Au_{n_k}\}$ converges to some $f \in X^*$:

$$u_{n_k} \rightharpoonup u,\ u \in X \ \&\ Au_{n_k} \rightharpoonup f,\ f \in X^*.$$

Need to show $f = b$. Let $v \in X$ be arbitrary, then, there exists $v_n \in X_n$ such that $v_n \to v$. From this, and weak convergence of $\{Au_{n_k}\}$ to $f \in X^*$ and boundedness: $\langle Au_n, v_n \rangle = \langle b, v_n \rangle \to \langle b, v \rangle$

and thus $\langle b, v \rangle = \langle f, v \rangle \quad \forall v \in X \quad \Rightarrow b = f.$

4) We have shown that $u_{n_k} \rightharpoonup u$ and $Au_{n_k} \rightharpoonup b$.

From Galerkin approximation and weak convergence of $\{u_{n_k}\}$

$$\langle Au_{n_k}, u_{n_k} \rangle = \langle b, u_{n_k} \rangle \to \langle b, u \rangle \quad \left(\limsup_{n\to\infty} \langle Au_n, u_n \rangle = \langle b, u \rangle\right)$$

From this, and $(M)_0$: $Au = b$

5) (1) and weak convergence proves last step □

<u>Corollary 2.16</u> Let X be separable, reflexive, Banach space and $A: X \to X^*$ meets <u>one</u> of the following conditions:

(i) A coercive, demicontinuous, bounded, & satisfies (S)

(ii) A coercive, pseudomonotone, & bounded

Then, statement of Th. 2.15 holds.

<u>Theorem 2.17 (Minty-Browder)</u>

Let X be reflexive Banach space and $A: X \to X^*$ monotone, radially continuous, and coercive. Then, A is surjective. The set of all solutions of $Au = b$, $b \in X^*$, is closed and convex. Additionally, function $N: \mathbb{R}_+ \to \mathbb{R}_+$ exists such that $\forall u \in X$

$$\|u\| \le N(\|Au\|).$$

Moreover, if A is strictly monotone then solution of $Au=b$ is unique for every right hand side $b \in X^*$

For studying existence of weak solutions of boundary value problems we use Browder & Leray-Lions:

Theorem 2.18 Let X be a reflexive Banach space. Let T be operator defined on X by values in X^* which is bounded, demicontinuous, and coercive. Let there be a bounded map ϕ from $X \times X$ to X^* such that

a) for each $u \in X$ $\phi(u,u) = Tu$;

b) $\forall u, v, h \in X$ and any sequence $\{t_n\}_{n=1}^{\infty} \in \mathbb{R}$ such that $t_n \to 0$ $\phi(u + t_n h, w) \to \phi(u, w)$

c) $\forall u, w \in X$ monotonicity in first term is met:
$$\langle \phi(u,u) - \phi(w,u), u - w \rangle \geq 0$$

d) for $u_n \rightharpoonup u$ and $\lim\limits_{n \to \infty} \langle \phi(u_n, u_n) - \phi(u, u_n), u_n - u \rangle = 0$;

then, for any $w \in X$
$$\phi(w, u_n) \to \phi(w, u)$$

e) if $w \in X$, $u_n \rightharpoonup u$, $\phi(w, u_n) \rightharpoonup z$; then,
$$\lim_{n \to \infty} \langle \phi(w, u_n), u_n \rangle = \langle z, u \rangle.$$

Then, the equation $Tu = f$ has at least one solution $u \in X$ for every $f \in X^*$.