Horner's Scheme

Numerical Mathematics

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Every polynomial, of degree n, of the form

$$p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$

can be written in the form

$$\begin{aligned} p(x) &= \underbrace{(\dots \underbrace{\underbrace{(\underbrace{(\underbrace{a_n}_{h_n(x)}\, x + a_{n-1})}_{h_{n-1}(x)}\, x + a_{n-2})}_{h_{n-2}(x)}\, x + \cdots + a_1)}_{h_1(x)}\, x + a_0}_{h_0(x)} \\ &= h_0(x), \end{aligned}$$

where

$$h_i(x) = \begin{cases} x h_{i-1}(x) + a_i & \text{if } i > 0, \\ a_n & \text{if } i = 0. \end{cases}$$

This gives us a recursive definition of the Horner's polynomials $h_i(x)$, $i = 0, \dots, n$, and allows use to the following algorithm to compute the polynomial.

Algorithm 1 (Horner's Scheme)**.** *For a polynomial* $p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$*, we can define the Horner's polynomials* $h_i(x)$*,* $i = 0, \dots, n$ *by:*

$h_n(x) = a_n$
for $i = n-1, n-2, \dots, 1, 0$ ***do***
  $h_i(x) = x h_{i+1}(x) + a_i$
end for
$p(x) = h_0$

Dividing polynomial by $(x - \alpha)$

We can use the Horner's scheme algorithm to divide a polynomial by $(x - \alpha)$. We first consider the following Theorem:

Theorem 2. *Let* f *and* g *be polynomials, where* $g \neq 0$*. Then, there exists polynomials* r *and* s *such that*

- $f = gs + r$*, and*
- *either* $r = 0$ *or* $\deg(r) < \deg(g)$*.*

The polynomials s *and* r *which satisfy these conditions are unique.*

Proof. See Theorem 4, page 128 of Hoffman & Kunze, *Linear Algebra*, Prentice Hall, 1971. □

This theorem proves that long division of a polynomial with real or complex coefficients is possible.

We now consider division of the polynomial by $(x-\alpha)$, this gives that

$$p(x) = (x-\alpha)q(x) + r(x),$$

where $q(x)$ is the polynomial that results from dividing $p(x)$ by $(x-\alpha)$ and $r(x)$ is the remainder of the division. From the previous theorem $r(x)$ must be a constant, as $r = 0$ or $1 = \deg(x-\alpha) > \deg(r) \implies \deg(r) = 0$ (constant). Then,

$$p(\alpha) = 0 + r(\alpha) \qquad \implies \qquad r(x) = p(\alpha).$$

From this we get that

$$\begin{aligned}
p(x) &= a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \\
&= (x-\alpha)\underbrace{(q_{n-1}x^{n-1} + q_{n-2}x^{n-2} + \cdots + q_q x + q_0)}_{q(x)} + p(\alpha) \\
&= p_{n-1}x^m + (q_{n-2} - \alpha q_{n-1})x^{n-1} + \cdots + (q_1 - \alpha q_2)x^2 - (q_0 - \alpha q_1)x - \alpha q_0 + p(\alpha).
\end{aligned}$$

By equating coefficients we get that

$$\begin{aligned}
a_n &= q_{n-1} & &\implies & q_{n-1} &= a_n = h_n(\alpha) \\
a_{n-1} &= q_{n-2} - \alpha q_{n-1} & &\implies & q_{n-2} &= \alpha q_{n-1} + a_{n-1} = \alpha h_n(\alpha) + a_{n-1} = h_{n-1}(\alpha) \\
a_{n-2} &= q_{n-3} - \alpha q_{n-2} & &\implies & q_{n-3} &= \alpha q_{n-2} + a_{n-2} = \alpha h_{n-1}(\alpha) + a_{n-2} = h_{n-2}(\alpha) \\
&\vdots & & & &\vdots \\
a_1 &= q_0 - \alpha q_1 & &\implies & q_0 &= \alpha q_1 + a_1 = \alpha h_2(\alpha) + a_1 = h_1(\alpha) \\
a_0 &= p(\alpha) - \alpha q_0.
\end{aligned}$$

Then, we get that

$$q(x) = \sum_{i=0}^{n-1} q_i x^i = \sum_{i=0}^{n-1} h_{i+1}(\alpha)x^i = \frac{p(x) - p(\alpha)}{x-\alpha}.$$

Therefore, we can compute the coefficients $q_i = h_{i+1}(x)$ of the polynomial $q(\alpha)$ and remainder $r(x) = p(\alpha)$ resulting from dividing $p(x)$ by $(x-\alpha)$. If α is a root of the polynomial p, then $p(\alpha) = 0$ clearly and roots of the polynomial q are also roots of the equation $p(x)$. This allows us to compute all roots of a polynomial, if we can find a root of any polynomial, for example by Newton's method.

Computing Derivative of Polynomial at Point α by Horner's Scheme

From the above we have that

$$p(\alpha) = (x-\alpha)q(x) + p(\alpha)$$

and, hence, we get that the derivative of p is

$$p'(\alpha) = q(x) + (x-\alpha)q'(x);$$

therefore, we can compute the derivative of p at α as

$$p'(\alpha) = q(\alpha).$$

So, applying the Horner's scheme to the polynomial q at α, we get the following algorithm for computing the derivative of the polynomial p at α:

Algorithm 3. *For a polynomial* $p(x) = a_0+a_1x+a_2x^2+\cdots+a_nx^n$, *we can compute the derivative at the point* α *by:*

$c_{n-1} = h_n(\alpha)$
for $i = n-2, n-3, \ldots, 1, 0$ ***do***
    $c_i = \alpha c_{i+1} + h_{i+1}(\alpha)$
end for
$q(\alpha) = p'(\alpha) = c_0$

where $h_i(x)$, $i = 0, \ldots, n$ *is computed by Horner's scheme.*

Exercises

1. Consider the polynomial

$$p(x) = 6x^4 - 8x^3 - 11x^2 - 3x + 18.$$

   (a) Use Horner's scheme to compute $h_i(2)$, $i = 0, \ldots, n$, and, hence, compute $p(2)$.

   (b) Use Horner's scheme to compute the polynomials $q(x)$ and $r(x)$, where

$$p(x) = (x-2)q(x) + r(x).$$

   (Divide $p(x)$ by $(x-2)$ using Horner's scheme).

   (c) Use Algorithm 3 to compute the derivative of $p(x)$ at $x = 2$.

2. Consider the polynomial

$$p(x) = 3x^4 - 22x^3 - 17x^2 - 6x + 22.$$

   (a) Use Horner's scheme to compute $h_i(-8)$, $i = 0, \ldots, n$, and, hence, compute $p(-8)$.

   (b) Use Horner's scheme to compute the polynomials $q(x)$ and $r(x)$, where

$$p(x) = (x+8)q(x) + r(x).$$

   (Divide $p(x)$ by $(x+8)$ using Horner's scheme).

   (c) Use Algorithm 3 to compute the derivative of $p(x)$ at $x = -8$.