Nonlinear Functional Analysis

Practicals

13th May 2020

Recap

Theorem 3.24. *Let $A : X \to X^*$ be a strictly monotone, coercive, potential operator, which satisfies condition (S); i.e.,*

$$[w_n \rightharpoonup w, \langle Aw_n - Aw, w_n - w\rangle \to 0] \quad \Longrightarrow \quad w_n \to w.$$

Then, $Au = f$ has a unique solution for each right hand side $f \in X^*$, to which the Ritz approximation $u_n \in X_n$ converges.

Theorem 3.25. *Let X be a real, separable, reflexive Banach space such that X and X^* are strictly convex. Suppose that the operator A satisfies the following properties:*

1. *A is strictly monotone, coercive, potential operator, satisfying condition (S).*
2. *A is bounded Lipschitz continuous; i.e., there exists an increasing function $M : R_+ \to R_+$ such that*

   $$\|Au - Av\| \leq M(R)\|u - v\|, \qquad R = \max(\|u\|, \|v\|), \qquad \forall u, v \in X.$$

   Furthermore, $M(R) \to +\infty$ as $R \to +\infty$.

Choose $a > 0$. For any $v_0 \in X$ construct the sequence $\{v_i\}$ defined by

$$v_{i+1} = v_i - t_i z_i, \qquad \mathcal{U} z_i = Av_i - b, \qquad i = 0, 1, \ldots,$$

where

$$t_i = \min\left\{1, \frac{2}{a + M(\|v_i\| + \|Av_i - b\|}\right\}.$$

Then, the sequence $\{v_i\}$ converges to the solution of the equation $Au = b$.

Theorem 3.26. *Let the assumptions of Theorem 3.25 be fulfilled. Choose any initial guess w_0 and positive number a. Construct the sequence $\{w_n\}$ by*

$$w_{n+1} = w_n - t_n z_n, \qquad \mathcal{U}_{n+1} z_n = A_{n+1} w_n - b_{n+1}, \qquad n = 0, 1, \ldots,$$

where

$$t_i = \min\left\{1, \frac{2}{a + M(\|w_n\| + \|Aw_n - b\|}\right\}.$$

Then, the sequence $\{w_n\}$ converges to the solution of the equation $Au = b$.

Exercises

1. Prove Theorem 3.24.

   Hint. Use the equivalence of the Ritz and Galerkin approximations, and apply Theorem 3.23.

2. Assume that in Theorem 3.25 the operator A is Lipschitz continuous; i.e. there exists a constant L for any $x, y \in X$ such that

$$\langle Ax - Ay, x - y\rangle \leq L\|x - y\|.$$

   Show that in the iterative method we can select

$$t_i = \frac{2}{a + L} \qquad \forall i.$$

   Hint. Look at step 1 of the proof of Theorem 3.25.

3. Assume that A is a bounded Lipschitz continuous, uniformly monotone, potential operator. Show that Theorem 3.25 holds.

4. Prove Theorem 3.26.

   Hint. In a similar manner to the proof of Theorem 3.25 prove the following steps:

   (a) $G(w_n) \geq G(w_{n+1})$ and $G(w_n) - G(w_{n+1}) \to 0$.

   (b) $G(w_0) \geq G(w_n) \geq \|w_n\| \left(\frac{1}{2} \left(\frac{\|w_n\|}{2} \right) - \left\| b - \frac{1}{2} A0 \right\| \right)$.

   (c) There exists $+\infty > R > 0$ such that $\|w_n\| \leq R$ for $n = 0, 1, \ldots$.

   (d) $z_n \to 0$ as $n \to +\infty$.

   (e) $w_n \rightharpoonup u$. Use the fact that as the sequence $\{w_n\}$ is bounded there exists a weakly convergent subsequence w_{n_k} which weakly converges to w. Then, for an arbitrary $x \in X$ construct the sequence $\{x_i\}$ such that $x_i \to x$. Then show that

$$\langle b - Ax, w - x\rangle \geq 0.$$

   Hence, as all monotone potential operators are demicontinuous by Lemma 2.7 (4) $Aw = b$ and, hence, $w = u$. Therefore, as all weakly convergent subsequences converge to u the whole sequence weakly converges to u.

   (f) $\langle Aw_n - Au, w_n - u\rangle \to 0$. Use Lemma 2.7 to show that the sequence $\{Aw_n\}$ is bounded; i.e., there exists a constant $R_2 < \infty$ such that $\|Aw_n\| \leq R_2$. To do this, prove that

$$\langle Aw_n, w_n\rangle = \langle \mathcal{U}z_n + b, w_n\rangle \leq R_1 \leq \infty, \qquad R_1 = R(\|z_n\| + \|b\|).$$

   Let u_n be the nth Ritz approximation; then,

$$\begin{aligned}\langle Aw_n - Au, w_n - u\rangle &= \langle Aw_n - Au, w_n - u_n\rangle + \langle Aw_n - Au, u_n - u\rangle \\ &= \langle \mathcal{U}z_n, w_n - u_n\rangle + \langle Aw_n - b, u_n - u\rangle \\ &\leq 2R\|z_n\| + (R_2 + \|b\|)\|u_n - u\| \to 0.\end{aligned}$$

   (g) Apply (S) to show that $w_n \to u$.