Nonlinear Functional Analysis

Practicals

13th May 2020

Dual Functionals

We shall look at the construction of a few dual functionals.

Example 1. Let $F \in X^*$. Then,

$$F^*(x^*) = \sup_{x \in X} \langle x^* - F, x \rangle.$$

It is known from (linear) functional analysis that there exists a x_0, $\|x_0\| = 1$ such that $\langle x^* - F, x_0 \rangle = \|x^* - F\|$. Then, for $x_1 = cx_0$, $c > 0$, $x^* \neq F$,

$$F^*(x^*) \geq \langle x^* - F, x_1 \rangle = c\|x^* - F\| \to \infty \text{ as } c \to \infty$$

and, thus,

$$F^*(x^*) = \begin{cases} +\infty & \text{if } x^* \neq F \\ F^*(x^*) = 0 & \text{if } x^* = F. \end{cases}$$

Example 2. Choose $\alpha > 1$ and $F(x) = \|x\|^\alpha$. Then,

$$F^*(x^*) = c\|x^*\|^{\alpha/(\alpha-1)}, \qquad c = (1-\alpha)\alpha^{-\alpha/(\alpha-1)}.$$

Proof.

$$F^*(x^*) \leq \sup_{x \in X} (\|x^*\|\|x\| - \|x\|^\alpha).$$

For $t \geq 0$ we define the function:

$$f(t) = \|x^*\|t - t^\alpha.$$

Since $f(0) = 0$ and $f(t) \to -\infty$ as $t \to +\infty$ then the function f achieves its maximum at the point t_0:

$$t_0 = \left(\frac{1}{\alpha}\|x^*\|\right)^{1/(\alpha-1)}, \qquad f(t_0) = c\|x^*\|^{\alpha/(\alpha-1)}, \qquad c = (\alpha-1)\alpha^{-\alpha/(\alpha-1)}$$

and, thus,

$$F^*(x^*) \leq c\|x^*\|^{\alpha/(\alpha-1)}.$$

We have proven that $F^*(x^*)$ is less than or equal to the desired definition, so we just need to show it is also greater than or equal to the definition. Construct $x_0 \in X$, $\|x_0\| = 1$ such that

$$\langle x^*, x_0 \rangle = \|x^*\|$$

and define

$$x_1 = kx_0, \qquad k = \|x^*\|^{1/(\alpha-1)} \left(\frac{1}{\alpha}\right)^{1/(\alpha-1)};$$

then,

$$F(x^*) \geq \langle x^*, x_1 \rangle - \|x_1\|^\alpha = c\|x^*\|^{\alpha/(\alpha-1)}, \qquad c = (\alpha-1)\alpha^{-\alpha/(\alpha-1)}.$$

□

We state the following two theorems for potential operators.

Theorem 3.20. *Let $A : X \to X^*$ be a strictly monotone, coercive, potential operator. Then, there exists an inverse operator A^{-1}, which is a strictly monotone potential operator. The functional F,*

$$F(x) = \int_0^1 \langle Atx, x\rangle \,\mathrm{d}t, \qquad x \in X,$$

is the potential of A and for any $x \in X$ and $x^* \in X$

$$\begin{aligned} F^*(x^*) &= F^*(0) + \int_0^1 \langle x^*, A^{-1}tx^x\rangle \,\mathrm{d}t, & F^*(0) = -F(A^{-1}0),\\ 0 &\le F(x) + F^*(x^*) - \langle x^*, x\rangle, \\ 0 &= F(x) + F^*(Ax) - \langle Ax, x\rangle, \end{aligned}$$

where F^* is the potential of A^{-1}.

Lemma 3.21. *Let $A : X \to X^*$ be a strictly monotone, coercive, potential operator with potential F. For any $f \in X^*$ there exists a unique solution $u \in X$ of $Au = f$ which minimises the potential of the problem $G = F - f$ and*

$$\begin{aligned} G(u) &\equiv F(u) - \langle f, u\rangle \\ &= \min_{v\in X} \left(\int_0^1 \langle Atv, v\rangle \,\mathrm{d}t - \langle f, v\rangle \right) \\ &= -\int_0^1 \langle f, A^{-1}tf\rangle \,\mathrm{d}t + \int_0^1 \langle AtA^{-1}0, A^{-1}0\rangle \,\mathrm{d}t. \end{aligned}$$

Exercises

1. Choose $F(x) = \|x\|$. Show that

$$F^*(x^*) = \begin{cases} 0, & \|x^*\| \le 1, \\ +\infty, & \|x^*\| > 1. \end{cases}$$

   Hint. Use the estimate that

$$F^*(x^*) \le \sup_{x\in X} (\|x^*\| - 1)\|x\|.$$

2. Prove Theorem 3.20 and Corollary 3.21.