1. $A \in \mathcal{L}(X)$, X real Hilbert space.

$$f_1(x) = (Ax, x) \qquad f_2(x) = \tfrac{1}{2}(Ax - z, Ax - z), \quad z \in X$$

$$(f_1'(x), h) = \lim_{t\to 0} \frac{1}{t}\big(f_1(x+th) - f_1(x)\big)$$

$$= \lim_{t\to 0} \frac{1}{t}\big((A(x+th), x+th) - (Ax, x)\big)$$

$$= \lim_{t\to 0} \frac{1}{t}\big((Ax,x) + t(Ah,x) + t(Ax,h) + t^2(Ah,h) - (Ax,x)\big)$$

$$= \lim_{t\to 0} \frac{1}{t}\big(t(Ah,x) + t(Ax,h) + t^2(Ah,h)\big)$$

$$= \lim_{t\to 0} \big((Ah,x) + (Ax,h) + t(Ah,h)\big)$$

$$= (Ah, x) + (Ax, h)$$

$$= (h, A^*x) + (Ax, h)$$

$$= ((A+A^*)(x), h) \qquad \forall h \in X$$

$$(f''(x), hg) = \lim_{t\to 0} \frac{1}{t}\big(f_1'(x+tg)h - f_1'(x)h\big)$$

$$= \lim_{t\to 0} \frac{1}{t}\big(((A+A^*)(x+tg), h) - ((A+A^*)(x), h)\big)$$

$$= \lim_{t\to 0} \frac{1}{t}\big((A+A^*)(tg), g\big)$$

$$= ((A+A^*)(g), h) = ((A+A^*)(h), g) \qquad \forall h,g \in X$$

$$f_2'(x) = \lim_{t\to 0} \frac{1}{2t}\Big[(A(x+th)-z, A(x+th)-z) - (Ax-z, Ax-z)\Big]$$

$$= \lim_{t\to 0} \frac{1}{2t}\Big[(Ax-z, Ax-z) + t(Ah, Ax-z) + t(Ax-z, Ah) + t^2(Ah, Ah) - (Ax-z, Ax-z)\Big]$$

$$= \lim_{t\to 0} \frac{1}{2t}\Big[t(Ah, Ax-z) + t(Ah, Ax-z) + t^2(Ah, Ah)\Big]$$

$$= \frac{1}{2}\Big[(Ah, Ax-z) + (Ax-z, Ah)\Big]$$

$$= (A^*(Ax-z), h)$$

$$(f_2''(x), hg) = \lim_{t \to 0} \frac{1}{t}\left(f_1'(x+tg)h - f_1'(x)h\right)$$
$$= \lim \frac{1}{t}\left[(A^*(A(x+tg)-z), h) - (A^*(Ax-z), h)\right]$$
$$= \lim \frac{1}{t}\left[(A^*Ax - A^*z, h) + (A^*A(tg), h) - (A^*(Ax-z), h)\right]$$
$$= \lim \frac{1}{t}(tA^*Ag, h)$$
$$= (A^*Ag, h) = (A^*Ah, g)$$

2. Show if A radially continuous

A potential operator $\Leftrightarrow \int_0^1 (Atx, x)dt - \int_0^1 (Aty, y)dt = \int_0^1 (A(y+t(x-y)), x-y)dt$ $\forall x, y \in X$

From lemma 3.3 proof (a) $\Rightarrow$ (b) still holds; i.e

A potential $\Rightarrow \int_0^1 (Atx, x)dt - \int_0^1 (Aty, y)dt = \int_0^1 (A(y+t(x-y)), x-y)dt$ $\forall x, y \in X$.

So just need to show " $\Leftarrow$ "

Show that $F(x) = \int_0^1 (Asx, x)ds$ is the potential of A

$$\lim_{t \to 0} \frac{F(u+tv) - F(v)}{t} = \lim_{t \to 0} \frac{1}{t}\left[\int_0^1 (A(s(u+tv)), u+tv)ds - \int_0^1 (Asu, u)ds\right.$$

Setting $x = u+tv$ & $y = u$, by left hand side of implication

$$\lim_{t \to 0} \frac{F(u+tv) - F(v)}{t} = \lim_{t \to 0} \frac{1}{t}\int_0^1 \langle A(u+stv), tv\rangle ds$$
$$= \lim_{t \to 0} \int_0^1 \langle A(u+stv), v\rangle ds$$

Let $\varphi_t'(s) = \langle A(u+stv), v\rangle$, then

$$\lim_{t \to 0} \frac{F(u+tv) - F(v)}{t} = \lim_{t \to 0} \int_0^1 \varphi_t'(s)\,ds = \lim_{t \to 0}\left[\varphi_t(1) - \varphi_t(0)\right]ds$$

By mean value theorem there exists a s_0 st $\varphi_t'(s_0) = \varphi_t(1) - \varphi_t(0)$

Therefore $\lim_{t \to 0} \frac{F(u+tv) - F(v)}{t} = \lim_{t \to 0} \varphi_t'(s_0) = \lim_{t \to 0} \langle A(u+s_0tv), v\rangle = \langle Au, v\rangle$

as A radially continuous

$\Rightarrow F$ potential of A $\Rightarrow A$ is a potential operator.

3. X real Hilbert space, $A \in \mathcal{L}(X)$. A is self-adjoint.

As A self-adjoint $\Rightarrow$ linear

Assume exists sequence $\{u_n\}$ weakly converges to u.

$$(Au_n, v) = (u_n, Av) \to (u, Av) = (Au, v)$$

$$\Rightarrow Au_n \rightharpoonup Au$$

$\Rightarrow$ A is weakly continuous $\Rightarrow$ demicontinuous $\Rightarrow$ radially cont.

Now consider

$$\int_0^1 (A(y + t(x-y)), x-y)\,dt$$

$$= \int_0^1 (Ay, x) - (Ay, y) + t(Ax, x) - t(Ax, y) - t(Ay, x) + t(Ay, y)\,dt$$

$$= (Ay, x) - (Ay, y) + \tfrac{1}{2}(Ax, x) - \underbrace{\tfrac{1}{2}(Ax, y)}_{= \frac{1}{2}(Ay, x)} - \underbrace{\tfrac{1}{2}(Ay, x)} + \tfrac{1}{2}(Ay, y)$$

as A self-adjoint.

$$= \tfrac{1}{2}(Ax, x) - \tfrac{1}{2}(Ay, y)$$

$$\int_0^1 (Atx, x)\,dt - \int_0^1 (Aty, y)\,dt = \tfrac{1}{2}(Ax, x) - \tfrac{1}{2}(Ay, y)$$

$$\Rightarrow \int_0^1 (Atx, x)\,dt - \int_0^1 (Aty, y)\,dt = \int_0^1 (A(y + t(x-y)), x-y)\,dt$$

So, as A demicontinuous, from lemma 3.3 A is a potential operator. From Lemma 3.2

$$F(x) = F(0) + \int_0^1 (Atx, x)\,dt = F(0) + (Ax, x)\int_0^1 t\,dt = \underbrace{F(0)}_{\text{Constant}} + \tfrac{1}{2}(Ax, x)$$

Let $F(0) = 0 \Rightarrow F(x) = \tfrac{1}{2}(Ax, x)$

From Exercise 1 $\quad \operatorname{grad} F(x) = \tfrac{1}{2}\operatorname{grad}(Ax, x)$

$$= \tfrac{1}{2}(A + A^*)(x)$$

$$= Ax \quad \text{as } A = A^*.$$

4. X^* strictly convex Banach space.

$\mathcal{U}: X \to X^*$ dualisation - strictly monotone, coercive, & demicontinuous

$$
\begin{aligned}
(\mathcal{U}y, y-x) &= (\mathcal{U}y, y) - (\mathcal{U}y, x) \\
&\geq \|y\|^2 - \|x\|\|y\| \\
&\geq \|y\|^2 - \tfrac{1}{2}(\|x\|^2 + \|y\|^2) \\
&= \tfrac{1}{2}\|y\|^2 - \tfrac{1}{2}\|x\|^2 \\
&\geq -\|x\|^2 + \|x\|\|y\| \\
&\geq (\mathcal{U}x, y-x)
\end{aligned}
$$

Select $y := x + th$, $h \in X$

$$t\langle \mathcal{U}x, h\rangle \leq \tfrac{1}{2}\|x+th\|^2 - \tfrac{1}{2}\|x\|^2 \leq t(\mathcal{U}x, y-x)$$

Divide by $t \neq 0$ & take limit

$$(\mathcal{U}x, h) \leq \lim_{t \to 0} \frac{1}{2t}\left[\|x+th\|^2 - \tfrac{1}{2}\|x\|^2\right] \leq \lim_{t\to 0} (\mathcal{U}(x+th), h)$$

$$\Rightarrow \langle \mathcal{U}x, h\rangle = \lim_{t\to 0} \frac{1}{t}\left[F(x+th) - F(x)\right]$$

where $F(x) = \tfrac{1}{2}\|x\|^2$

$\Rightarrow F$ potential for operator $\mathcal{U}$.

5. $F(x) = \|x\|^{\alpha+1}$, $\alpha > 0$ $\qquad$ $\|\cdot\|$ - Gateaux differentiable at every point

From definition of Gateaux-derivative for $x \neq 0$

$$\langle \operatorname{grad} F(x), h \rangle = (\alpha+1) \|x\|^{\alpha} \langle \operatorname{grad} \|x\|, h \rangle$$

$$Ax \equiv \operatorname{grad} F(x) = (\alpha+1) \|x\|^{\alpha} \operatorname{grad} \|x\| \in X \to X^*$$

. Thus, A potential operator with potential F.

A monotone:

Use fact that $\langle \operatorname{grad}\|x\|, x \rangle = \|x\|$ and $\|\operatorname{grad}\|x\|\| = 1$

$$(Ax - Ay, x-y) = \langle Ax, x \rangle + \langle Ay, y \rangle - \langle Ax, y \rangle - \langle Ay, x \rangle$$

$$\geq (\alpha+1) \left[\|x\|^{\alpha} \langle \operatorname{grad}\|x\|, x \rangle + \|y\|^{\alpha} \langle \operatorname{grad}\|y\|, y \rangle - \| \|x\|^{\alpha} \operatorname{grad}\|x\| \| \|y\| - \| \|y\|^{\alpha} \operatorname{grad}\|y\| \| \|x\| \right]$$

$$\geq (\alpha+1) \left[\|x\|^{\alpha+1} + \|y\|^{\alpha+1} - \|x\|^{\alpha}\|y\| - \|y\|^{\alpha}\|x\| \right]$$

$$= (\alpha+1)(\|x\| - \|y\|)(\|x\|^{\alpha} - \|y\|^{\alpha}) \geq 0$$

A coercive:

$$\langle Ax, x \rangle = (\alpha+1) \|x\|^{\alpha} \langle \operatorname{grad}\|x\|, x \rangle = (\alpha+1) \|x\|^{\alpha+1}, \; x \in X$$

$$\lim_{\|x\| \to \infty} \frac{\langle Ax, x \rangle}{\|x\|} = \lim_{\|x\| \to \infty} \frac{(\alpha+1) \|x\|^{\alpha} \langle \operatorname{grad}\|x\|, \|x\| \rangle}{\|x\|}$$

$$= \lim_{\|x\| \to \infty} \frac{(\alpha+1) \|x\|^{\alpha} \|x\|}{\|x\|}$$

$$= \lim_{\|x\| \to \infty} (\alpha+1) \|x\|^{\alpha} = +\infty$$