Nonlinear Functional Analysis

Practicals

30th April 2020

Gâteaux Derivatives

We are going to recap some properties of Gâteaux derivatives, without proof.

Definition 1. *Let X and Y be normed linear spaces and $A : X \to Y$ is a generally nonlinear operator. Let $x \in X$; then, if there exists the limit*

$$\lim_{t \to 0} \frac{1}{t} \left(A(x + th) - Ax \right) \equiv VA(x, h), \qquad \forall h \in X,$$

we say that A is weakly (Gâteaux) differentiable at the point x. In this case $VA(x, h)$ is called the variation or Gâtaeux differential *of the operator A at the point x in the direction h.*

From this definition it immediately follows that $VA(x, h)$ is homogeneous with respect to h; i.e.,

$$VA(x, \alpha h) = \alpha VA(x, h).$$

However, it may not be a linear functional in h, i.e.,

$$VA(x, h + \ell) = VA(x, h) + VA(c, \ell)$$

may not hold. If it is does, we denote the variation as $DA(x, h)$. In addition if this operator is continuous in h then it is called a *weak derivative* or *Gâteaux derivative* of the operator A at the point x in the direction h. In this case we denote the operator as A'; clearly,

$$A' : X \to \mathcal{L}(X, Y).$$

Theorem 2. *Suppose that the functional f is Gâteaux differentiable at each point in the convex subset Ω of the linear space X. Then, for any points $x, x + h \in \Omega$ there exists a $\tau \in [0, 1]$ such that*

$$f(x + h) - f(x) = Df(x + \tau h, h).$$

Corollary 3. *Let X be a normed space and the functional f have at each point $x \in X$ the Gâteaux derivative $f'(x) \in X^*$. Then,*

$$f(x + h) - f(x) = \langle f'(x + \tau h), h \rangle \qquad \forall x, h \in X,$$

for $0 < \tau < 1$.

Definition 4. *Let f be a nonlinear functional defined on a normed linear space X. If there exists a Gâteaux derivative f' at the point x, then we can call this the* gradient *of the functional f; i.e.,* $\operatorname{grad} f(x) \equiv f'(x)$*. This is a continuous linear functional over X: $f'(x) \in X^*$.*

Lemma 5. *Suppose that the norm in a real normed space X is Gâteaux-differentiable at every non-zero point $x \in X$ and $D(\|\cdot\|, h)$ is a linear functional in h for every $x \neq 0$. Then,*

1. *The Gâteaux differential $D(\|\cdot\|, h)$ is a continuous linear functional with respect to the variable h and, thus, is a Gâteaux derivative (the gradient of the norm); i.e, $D(\|x\|, h) = \operatorname{grad}\|x\|$ for $x \neq 0$.*

2. *For each $x \neq 0$ and $\alpha \neq 0$*

$$\|\operatorname{grad}\|x\|\| = 1, \qquad \langle \operatorname{grad}\|x\|, x \rangle = \|x\|, \qquad \operatorname{grad}\|\alpha x\| = \operatorname{sign}\alpha \operatorname{grad}\|x\|.$$

Recap

Lemma 3.3. *Let X be a reflexive Banach space and the operator $A : X \to X^*$ be demicontinuous. Then the following statements are equivalent:*

a) A is a potential operator.

b) For any $x, y \in X$

$$\int_0^1 \langle Atx, x\rangle \, \mathrm{d}t - \int_0^1 \langle Aty, y\rangle \, \mathrm{d}t = \int_0^1 \langle A(y + t(x-y)), x - y\rangle \, \mathrm{d}t.$$

c) For any $x, y \in X$ and any continuously differentiable function $u : [0,1] \to X$, such that $u(0) = x$ and $u(1) = y$,

$$\int_0^1 \langle Atx, x\rangle \, \mathrm{d}t - \int_0^1 \langle Aty, y\rangle \, \mathrm{d}t = \int_0^1 \langle Au(t), u'(t)\rangle \, \mathrm{d}t.$$

Definition 6. *The mapping $\mathcal{U} : X \to X^*$, where X is a Banach (or normed) space, is called a dualisation if for any element $x \in X$*

$$\|\mathcal{U}(x)\| = \|x\|, \qquad \langle \mathcal{U}(x), x\rangle = \|\mathcal{U}(x)\|\|x\| = \|x\|^2.$$

Exercises

1. Let X be a real Hilbert space and $A \in \mathcal{L}(X)$. Calculate the first and second Gateaux derivative of the functionals

$$f_1(x) = (Ax, x) \qquad \text{and} \qquad f_2(x) = (Ax - z, Ax - z), z \in X.$$

   Show that it applies that

$$f_1'(x) = (A + A^*)(x), \qquad f_1''(x) = A + A^*, \qquad f_2'(x) = A^*(Ax - z), \qquad f_2''(x) = AA^*,$$

   where A^* is a dual operator; i.e.,

$$\begin{aligned} f_1'(x) &= ((A + A^*)(x), h), \qquad & f_1''(x)hg &= ((A + A^*)h, g) = f_1''(x)gh, \\ f_2'(x) &= (A^*(Ax - z), h), \qquad & f_2''(x)hg &= (AA^*h, g) = f_2''(x)gh. \end{aligned}$$

2. Show that a) and b) of Lemma 3.3 are equivalent if the operator $A : X \to X^*$ is radially continuous instead of demicontinuous.

3. Let X be a real Hilbert space with operator $A \in \mathcal{L}(X)$. Prove that this operator is a potential operator if it is self-adjoint. Construct the matching potential.

   Hint. Use the criteria in Lemma 3.3 and Lemma 3.4, plus Exercise 1. To construct the potential use Lemma 3.2 and the continuity of the operator A. Show that the potential F of this operator (see Exercise 1) is of the form $F(x) = \frac{1}{2}(Ax, x)$ for all $x \in X$.

4. Let the space X^* be strictly convex, where X is a Banach space. Then, there exists a dualisation $\mathcal{U}$ (see Definition 6). If the space X is additionally reflexive, then the dualisation $\mathcal{U} : X \to X^*$ is strictly monotone, coercive, and demicontinuous (and, thus, radially continuous and hemicontinuous). Show that the operator $\mathcal{U}$ is a potential operator with potential

$$F(x) = \frac{1}{2}\|x\|^2, \qquad x \in X.$$

Hint. Show that

$$\langle \mathcal{U}y, y - x \rangle \geq \langle \mathcal{U}x, y - x \rangle;$$

then, substitute $y := x + th$, $h \in X$ into this inequality, use radially continuous property, and take the limit.

5. Let X be a real normed space. Assume that the norm $\|\cdot\|$ is Gâteaux differentiable at every non-zero point $x \in X$. Then, define

$$F(x) = \|x\|^{\alpha+1}, \qquad x \in X,$$

and show that

$$Ax \equiv \operatorname{grad} F(x), \qquad x \in X,$$

is a potential operator with potential F. Furthermore, show that A is monotone and coercive.

Hint. Use Lemma 5 to define Ax.